

9. CONTINUITY AND DIFFERENTIABILITY

FUNCTIONS

Real Functions

Let R be the set of all real numbers, and let X and Y be any two nonempty subsets of R . Then, a rule f which associates to each $x \in X$, a unique real number $f(x) \in Y$ is called a real function from X to Y and we write, $f : X \rightarrow Y$.

$f(x)$ is called the *image* of x or the *value of the function at x* . The sets X and Y are respectively known as the *domain* and the *codomain* of f .

Also, the set $\{f(x) : x \in X\}$ is called the *range* of f .

Clearly, $\text{range}(f) \subseteq Y$.

However, if $\text{range}(f) = Y$, we say that f is an *onto* function; otherwise f is said to be an *into* function.

If two or more than two elements in X have the same image in Y then f is said to be a *many-one* function.

On the other hand, if different elements in X have different images in Y , we say that f is *one-one*.

Clearly, f is one-one $\Leftrightarrow [f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$.

A one-one onto function is called a *one-to-one correspondence*.

REMARK 1 Sometimes a function is described only by a formula and the domain of the function is not explicitly stated. In such cases, the domain of the function is the set of all those real numbers for which the formula is meaningful.

REMARK 2 Usually the domain of a real function is an *interval*. For any two real numbers a and b , where $a < b$, we define

- (i) closed interval $[a, b] = \{x \in R : a \leq x \leq b\}$
 - (ii) open interval $]a, b[= \{x \in R : a < x < b\}$
 - (iii) right-half open interval $[a, b[= \{x \in R : a \leq x < b\}$
 - (iv) left-half open interval $]a, b] = \{x \in R : a < x \leq b\}$
 - (v) $]a, \infty[= \{x \in R : x > a\}$
 - (vi) $[a, \infty[= \{x \in R : x \geq a\}$
 - (vii) $] -\infty, a[= \{x \in R : x < a\}$
 - (viii) $] -\infty, a] = \{x \in R : x \leq a\}$
- Sometimes, we write, $R =] -\infty, \infty[$.

Some Important Functions

1. CONSTANT FUNCTION Let c be a fixed real number. Then, the function defined by $f(x) = c$ for all $x \in R$ is called a constant function c .

Clearly, $\text{dom}(f) = R$ and $\text{range}(f) = \{c\}$.

2. IDENTITY FUNCTION The function defined by $f(x) = x$ for all $x \in R$ is called the identity function.

Clearly, its domain is R and its range is R .

3. MODULUS FUNCTION The function defined by

$$f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$

is called the modulus function.

Since the modulus of every real number is a unique non-negative real number, so $\text{dom } f(x) = R$.

Since $|x|$ is either 0 or a positive real number, we have

$$\text{range}(f) = \{|x| : x \in R\} = \text{set of non-negative real numbers.}$$

4. RECIPROCAL FUNCTION The function defined by $f(x) = \frac{1}{x}$ is called the reciprocal function.

Clearly, $\frac{1}{x}$ is not defined when $x = 0$.

$$\therefore \text{dom}(f) = R - \{0\}.$$

$$\text{Also, } y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}.$$

Clearly, x is defined for all real numbers y except when $y = 0$.

$$\therefore \text{range}(f) = R - \{0\}.$$

5. SIGNUM FUNCTION This function is defined by $f(x) = \begin{cases} \frac{|x|}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0. \end{cases}$

$$\text{Thus, we have } f(x) = \begin{cases} 1, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -1, & \text{when } x < 0. \end{cases}$$

Clearly, its domain is R and $\text{range} = \{-1, 0, 1\}$.

6. SQUARE-ROOT FUNCTION Let $f(x) = +\sqrt{x}$.

We know that the negative real numbers do not have real square roots. So, $f(x)$ is not defined when x is a negative real number.

$$\therefore \text{dom}(f) = \text{set of all non-negative real numbers} = [0, \infty[.$$

$$\text{Clearly, } \text{range}(f) = \{+\sqrt{x} : x \in [0, \infty[\} = [0, \infty[.$$

7. STEP FUNCTION OR THE GREATEST INTEGER FUNCTION

If $x \in R$ then $[x]$ is defined as the greatest integer not exceeding x .

For example, we have

$$[2.01] = 2; [2.9] = 2; [-1.3] = -2; [3] = 3 \text{ and } [-1] = -1, \text{ etc.}$$

Now, if we consider $f(x) = [x]$ then clearly for each $x \in R$, $[x]$ is defined.

So $\text{dom}(f) = R$.

By definition, $[x]$ is an integer.

So, $\text{range}(f) = \{[x] : x \in R\} = \text{set of all integers}$.

EXAMPLE Find a set of all real numbers x such that $[x] = 2$.

SOLUTION Clearly, for all x such that $2 \leq x < 3$, we have $f(x) = [x] = 2$.

$\therefore$ required set $= \{x \in R : 2 \leq x < 3\} = [2, 3[$.

8. SMALLEST INTEGER FUNCTION (OR CEILING FUNCTION) For any real number x , we define $\lceil x \rceil$ as the smallest integer greater than or equal to x .

For example,

$$\lceil 6.3 \rceil = 7, \lceil 7.01 \rceil = 8, \lceil -6.1 \rceil = -6, \lceil -2.9 \rceil = -2, \lceil -3 \rceil = -3, \lceil 5 \rceil = 5.$$

The function $f : R \rightarrow R : f(x) = \lceil x \rceil$, $x \in R$ is called the smallest integer function or ceiling function.

Clearly, $\text{domain}(f) = R$ and $\text{range}(f) = I$.

9. POLYNOMIAL FUNCTION A function of the form

$$p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n,$$

where $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ are real numbers, $a_0 \neq 0$

and n is a non-negative integer, is called a polynomial function of degree n .

Polynomials of degree 1, 2, 3 and 4 are respectively called *linear*, *quadratic*, *cubic* and *biquadratic* polynomials.

Thus, (i) $f(x) = ax + b$, $a \neq 0$, is a linear polynomial.

(ii) $f(x) = ax^2 + bx + c$, $a \neq 0$, is a quadratic polynomial.

(iii) $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, is a cubic polynomial.

10. RATIONAL FUNCTION A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function.

Thus, $f(x) = \left(\frac{x^2 + 1}{x^3 - 2x + 5} \right)$ is a rational function, where $x^3 - 2x + 5 \neq 0$.

Graphs

Graph of a Function

For a given function $f(x)$, the aggregate of the points $\{x, f(x)\}$ is called the graph or the curve representing the function.

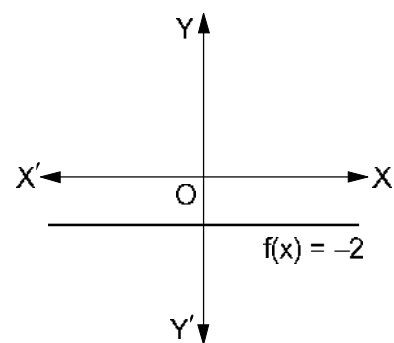
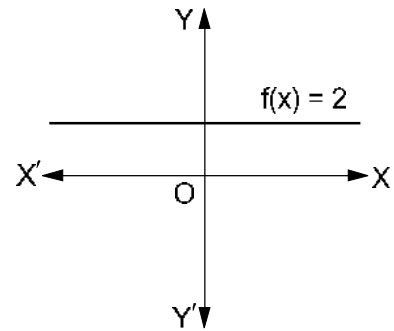
In practice, we plot some of the points and join them freehand to obtain the graph.

EXAMPLE 1 Draw the graphs of the constant functions

(i) $f(x) = 2$ (ii) $f(x) = 0$ (iii) $f(x) = -2$

SOLUTION

- (i) When $f(x) = 2$ for all $x \in \mathbb{R}$, some of the points on the graph may be taken as $(0, 2), (-1, 2), (1, 2), (-2, 2), (2, 2)$, etc. Joining these points, we obtain a line $y = 2$, drawn parallel to the x -axis at a distance of 2 units from it, as the required graph.
- (ii) The graph of the function $f(x) = 0$ is the line $y = 0$, i.e., the x -axis.
- (iii) The graph of the function $f(x) = -2$ is the line $y = -2$, drawn parallel to the x -axis at a distance of 2 units below the x -axis.



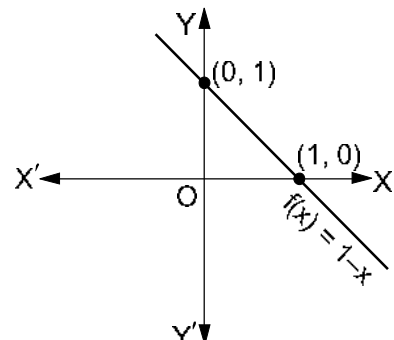
EXAMPLE 2 Draw the graphs of the linear functions

(i) $f(x) = 1 - x$ (ii) $f(x) = 2x + 1$

SOLUTION

- (i) When $f(x) = 1 - x$, some of the points on the graph are $(0, 1), (1, 0), (2, -1), (-1, 2)$, etc.

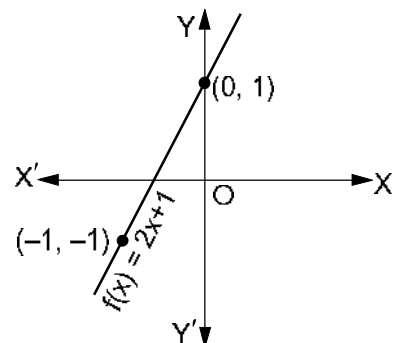
Joining these points, we get a line as the graph of the function.



- (ii) Let $f(x) = 2x + 1$.

Some of the points on the graph are $(0, 1), (1, 3), (-1, -1), (2, 5)$, etc.

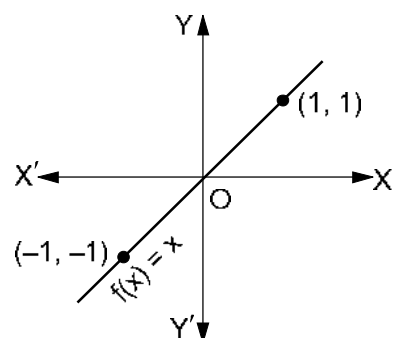
Joining these points, we obtain a line as the graph.



REMARK The graph of a linear function is a straight line.

EXAMPLE 3 Draw the graph of the identity function $f(x) = x$.

SOLUTION $f(x) = x$ is clearly a linear function whose graph must be a line. Plotting the points $(0, 0), (1, 1), (-1, -1)$, etc., and joining them, we get the required graph.



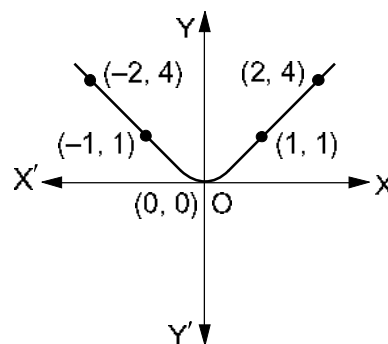
EXAMPLE 4 Draw the graphs of the polynomial functions

(i) $f(x) = x^2$ (ii) $f(x) = 1 - x^2$

(iii) $f(x) = x^3 - x$

SOLUTION

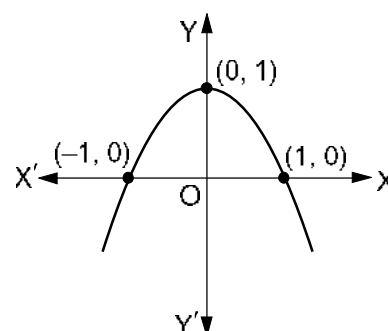
(i) The function $f(x) = x^2$ is a quadratic function. Some of the points on the graph are $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(-2, 4)$, $(2, 4)$, $(-3, 9)$, $(3, 9)$, etc. Joining these points, we get a parabola as the graph.



(ii) $f(x) = 1 - x^2$ is also a quadratic function. Some of the points on the graph are

$(0, 1)$, $(1, 0)$, $(-1, 0)$, $(2, -3)$, $(-2, -3)$, $(-3, -8)$, $(3, -8)$, etc.

Joining these points, we obtain a parabola as its graph.

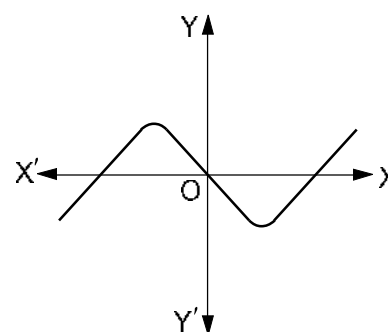


(iii) Let $f(x) = x^3 - x$.

This is a cubic function.

Some of the points on the graph are $(0, 0)$, $(-1, 0)$, $(1, 0)$, $(-0.5, 0.375)$, $(0.5, -0.375)$, $(2, 6)$, $(-2, -6)$, etc.

Joining these points, we obtain the required graph.



REMARK It may be observed here that whenever (x, y) is a point on the graph then

$(-x, -y)$ is also a point on the graph. So, the graph is symmetrical about the origin.

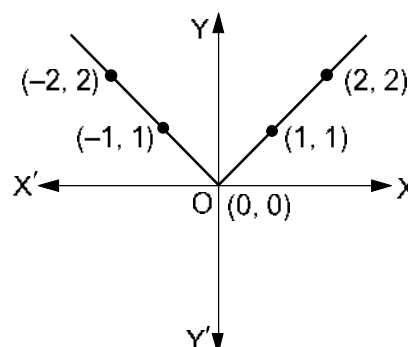
This is an important property possessed by the graph of an odd function.

EXAMPLE 5 Draw the graph of the modulus function $f(x) = |x|$.

SOLUTION
$$f(x) = |x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

Some of the points on the graph are $(0, 0)$, $(-1, 1)$, $(-2, 2)$, $(1, 1)$, $(2, 2)$, etc.

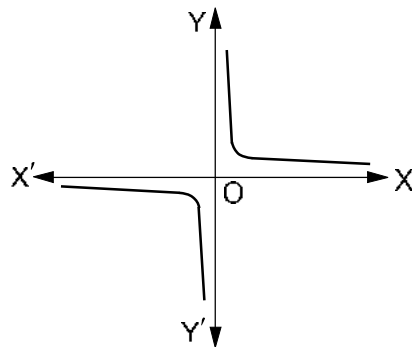
Joining these points, we get the required graph.



REMARK It may be observed here that whenever (x, y) is a point on the graph, then $(-x, y)$ is also a point of it. Thus, *the graph is symmetrical about the y -axis*. This is an important property possessed by the graphs of even functions.

EXAMPLE 6 Draw the graph of the reciprocal function $f(x) = \frac{1}{x}$.

SOLUTION Clearly, $f(x) = \frac{1}{x}$ is not defined at $x = 0$. Some points on the graph are $(1, 1)$, $(-1, -1)$, $(1/2, 2)$, $(2, 1/2)$, $(-1/2, -2)$, $(-2, -1/2)$, $(1/3, 3)$, $(-1/3, -3)$, $(3, 1/3)$, etc. Joining these points, we get the required graph. Since $f(x) = \frac{1}{x}$ is an odd function, it is symmetrical about the origin.



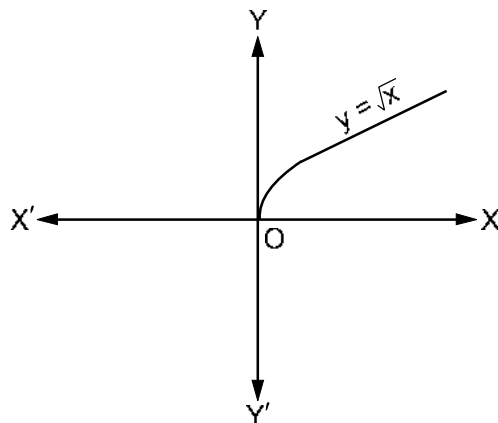
EXAMPLE 7 Draw the graph of the square-root function $f(x) = \sqrt{x}$.

SOLUTION Let f be a real-valued function which associates to each non-negative real number x , its non-negative square root.

Then, $f : R_0^+ \rightarrow R_0^+ : f(x) = \sqrt{x}$, is called the square-root function.

Domain $f(x) = R_0^+$, and range $(f) = R_0^+$.

Some of the points on the graph are $(0, 0)$, $(1, 1)$, $(2, 1.4)$, $(3, 1.7)$, $(4, 2)$, $(5, 2.2)$, etc. Joining these points, we get the required graph.



EXAMPLE 8 Draw the graph of the rational function $f(x) = \frac{x^2 - 1}{x - 1}$.

SOLUTION Let $f(x) = \frac{x^2 - 1}{x - 1}$.

Now, $f(1) = \frac{0}{0}$, which is meaningless.

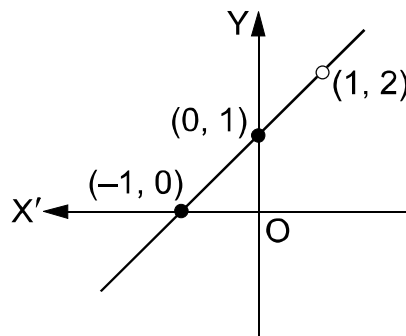
So, the function is not defined at $x = 1$.

Also, when $x \neq 1$, we have

$$f(x) = \left(\frac{x^2 - 1}{x - 1} \right) = (x + 1).$$

This being a linear function, its graph is a straight line.

Some of the points on the graph are $(0, 1)$, $(-1, 0)$, $(2, 3)$, $(3, 4)$, $(-2, -1)$, $(-3, -2)$, etc.



Joining these points, we obtain the required graph. Clearly, the point $(1, 2)$ does not lie on the graph. So, *it is a broken graph*, and we shall say that the given function is discontinuous at $x = 1$.

EXAMPLE 9 Draw the graph of the step function $f(x) = [x]$.

SOLUTION As the definition of the function indicates,

for all x such that $-2 \leq x < -1$,

we have $f(x) = -2$;

for all x such that $-1 \leq x < 0$,

we have $f(x) = -1$;

for all x such that $0 \leq x < 1$,

we have $f(x) = 0$;

for all x such that $1 \leq x < 2$, we have $f(x) = 1$, and so on,

$$\text{i.e., } f(x) = \begin{cases} -2 & \text{when } x \in [-2, -1[\\ -1 & \text{when } x \in [-1, 0[\\ 0 & \text{when } x \in [0, 1[\\ 1 & \text{when } x \in [1, 2[\\ \text{and so on.} \end{cases}$$

Clearly, the function jumps at the points $(-1, -2)$, $(0, -1)$, $(1, 0)$, $(2, 1)$, etc.

In other words, *the given function is discontinuous at each integral value of x .*

EXAMPLE 10 Draw the graph of the smallest integer function $f(x) = \lceil x \rceil$.

SOLUTION As the definition of the function suggests,

for all x such that $-3 < x \leq -2$, we have $f(x) = -2$;

for all x such that $-2 < x \leq -1$, we have $f(x) = -1$;

for all x such that $-1 < x \leq 0$, we have $f(x) = 0$;

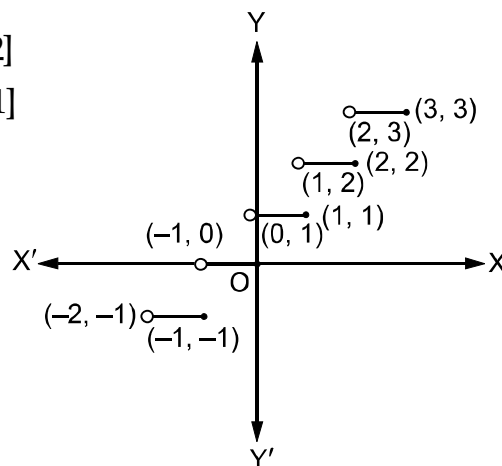
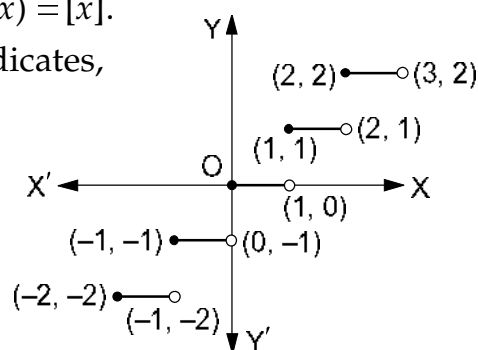
for all x such that $0 < x \leq 1$, we have $f(x) = 1$;

and so on.

$$\text{i.e., } f(x) = \begin{cases} -2 & \text{when } x \in]-3, -2] \\ -1 & \text{when } x \in]-2, -1] \\ 0 & \text{when } x \in]-1, 0] \\ 1 & \text{when } x \in]0, 1] \\ 2 & \text{when } x \in]1, 2] \\ 3 & \text{when } x \in]2, 3] \\ \text{and so on.} \end{cases}$$

Plotting these points, we can get the required graph. The function

jumps at the points $(-2, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$, etc., or is *discontinuous at each integral value of x .*



EXAMPLE 11 Draw the graph of the signum function $f(x) = \begin{cases} \frac{|x|}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$

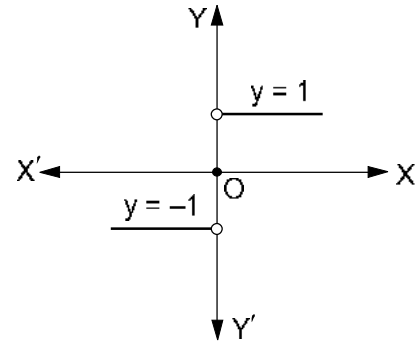
SOLUTION Clearly, $(0, 0)$ is a point on the graph. Now, when $x > 0$, we have $|x| = x$, and so in this case, we have, $f(x) = 1$, i.e., $f(x) = 1$ for all values of $x > 0$.

And, when $x < 0$, we have $|x| = -x$ and therefore,

$$f(x) = -1 \text{ for all values of } x < 0.$$

Hence the graph may be drawn, as shown in the adjoining figure.

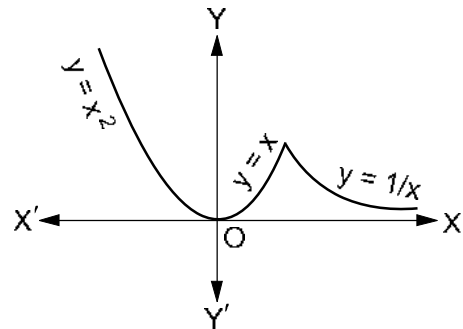
Clearly, the function is broken (i.e., it is discontinuous) at each of the points $x = -1, 0$ and 1 .



EXAMPLE 12 Draw the graph of the function $f(x) = \begin{cases} x^2, & \text{when } x < 0 \\ x, & \text{when } 0 \leq x \leq 1 \\ 1/x, & \text{when } 1 \leq x < \infty. \end{cases}$

SOLUTION Here, the graph consists of three parts. Some of the points of the graph are $(-3, 9)$, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{4}, \frac{3}{4})$, $(1, 1)$, $(2, \frac{1}{2})$, $(3, \frac{1}{3})$, etc.

And, the graph may now be drawn, as shown in the adjoining figure.



EXAMPLE 13 Draw the graph of the function $f(x) = |x| + |x - 1|$.

SOLUTION Let us consider the following cases.

Case I When $x < 0$

In this case, $(x - 1) < 0$.

$$\therefore |x| = -x \text{ and } |x - 1| = -(x - 1) = 1 - x.$$

$$\text{Consequently, } |x| + |x - 1| = -x + 1 - x = 1 - 2x.$$

Case II When $0 \leq x \leq 1$

$$\text{In this case, } |x| = x \text{ and } |x - 1| = -(x - 1) = 1 - x.$$

$$\therefore |x| + |x - 1| = x + 1 - x = 1.$$

Case III When $x > 1$

$$\text{In this case, } |x| = x \text{ and } |x - 1| = x - 1.$$

$$\therefore |x| + |x - 1| = x + (x - 1) = (2x - 1).$$

Thus, we may define the above function as

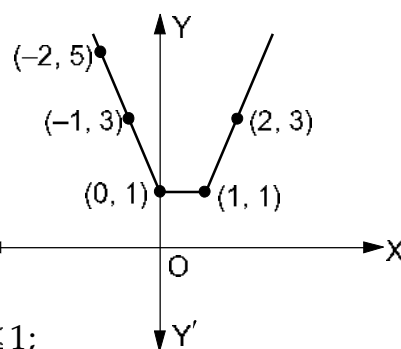
$$f(x) = \begin{cases} 1-2x, & \text{when } x < 0 \\ 1, & \text{when } 0 \leq x \leq 1 \\ 2x-1, & \text{when } x > 1. \end{cases}$$

So, we have

- (i) a linear function $1 - 2x$ when $x < 0$;
- (ii) a constant function 1 when $0 \leq x \leq 1$;
- (iii) a linear function $2x - 1$ when $x > 1$.

The corresponding points on these parts of the graph are $(-1, 3), (-2, 5), (0, 1), (1, 1), (2, 3), (3, 5)$, etc.

Joining these points, we obtain the graph as shown.



EXAMPLE 14 Draw the graph of the exponential function:

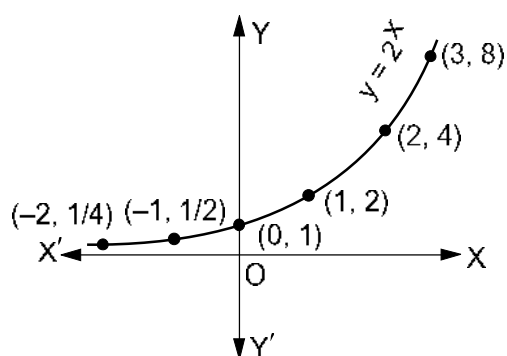
$$(i) f(x) = 2^x \quad (ii) f(x) = \left(\frac{1}{3}\right)^x$$

SOLUTION

- (i) Let $f(x) = 2^x$.

Some of the points on the graph are

$$(0, 1), (1, 2), (2, 4), (3, 8), \left(-1, \frac{1}{2}\right), \left(-2, \frac{1}{4}\right), \left(-3, \frac{1}{8}\right), \text{ etc.}$$

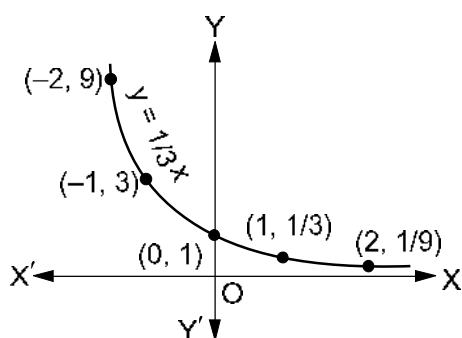


And so the graph takes the form, shown in the adjoining figure.

It may be observed here that the *given function is strictly increasing*. Also, as the value of x decreases, the corresponding value of the function decreases, and therefore, on the left-hand side of the y -axis, the curve comes closer and closer to the x -axis.

REMARK This is the case of the exponential function a^x , where $a > 1$.

$$(ii) \text{ Let } f(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x}.$$



Some of the points on the graph are $(-2, 9), (-1, 3), (0, 1), \left(1, \frac{1}{3}\right), \left(2, \frac{1}{9}\right)$, etc.

Joining these points, we obtain the graph as shown. It follows from the graph that *the given function is strictly decreasing*.

On the RHS of the y -axis, the curve comes closer and closer to the x -axis.

REMARK This is the case of the exponential function a^x , where $0 < a < 1$.

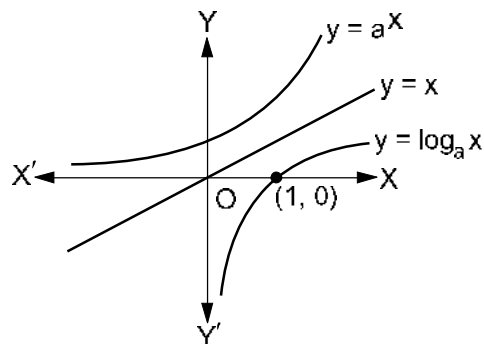
EXAMPLE 15 Draw the graphs of the logarithmic functions(i) $\log_a x$, when $a > 1$ (ii) $\log_a x$, when $0 < a < 1$.**SOLUTION**

(i) We know that when $a > 1$, the function a^x is strictly increasing, i.e., different values of x give different values of a^x . Also, the range of this function is R .

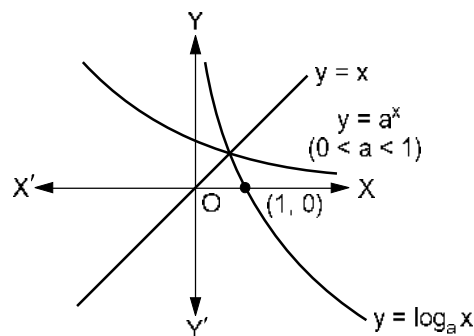
So, the function $f(x) = a^x$ is one-one onto and therefore invertible.

Its graph is of the form shown in the adjoining figure. The graph of the function $g(x) = \log_a x$ is the reflection of the graph of $f(x) = a^x$ in the line $y = x$.

It may be noted that the graph passes through $(1, 0)$.



(ii) We know that when $0 < a < 1$, the function a^x is strictly decreasing, i.e., different values of x give different values of a^x . So, the function is one-one. Also, its range is R . So, it is onto. Thus, the function a^x is invertible.

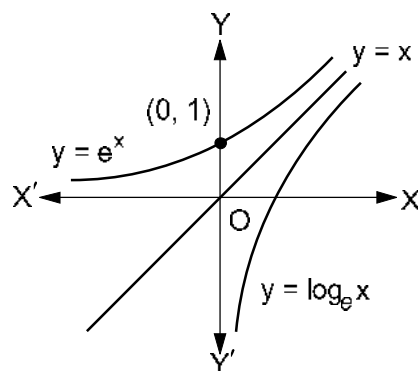


Its inverse is the log function reflected by the line $y = x$, as shown. The graph clearly passes through $(1, 0)$.

EXAMPLE 16 On the same scale draw the graphs of e^x and $\log_e x$.

SOLUTION Since e lies between 2 and 3, it follows that $e > 1$. So, it is a particular case of a^x , where $a > 1$. The function e^x is strictly increasing. Also, its range is R . So, the function is one-one onto and therefore invertible.

The graph passes through the point $(0, 1)$ and comes closer and closer to the x -axis and the values of x decrease. Thus, the graph of e^x may be drawn as shown in the figure. Its reflection in the line $y = x$ gives the graph of $\log_e x$.

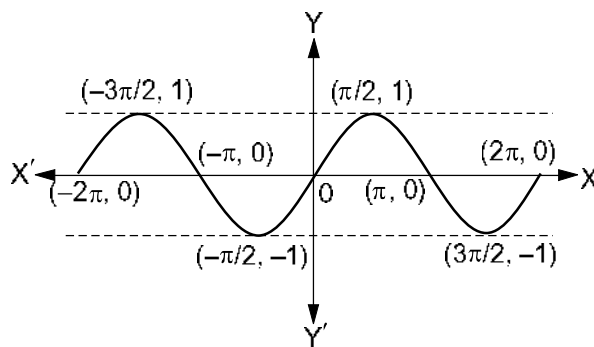


Graphs of Trigonometric Functions

1. GRAPH OF A SINE FUNCTION We know that a sine function is periodic with period 2π . So, we have to sketch the graph in the interval $[0, 2\pi]$ and then we may complete the graph by repeating it over intervals, each of length 2π . We first draw it in the interval $[0, \pi/2]$. We know that it is strictly increasing in this interval. Also, we may use the table, given below.

x	0°	30°	45°	60°	90°
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = 0.707$	$\frac{\sqrt{3}}{2} = 0.87$	1

Thus, we may draw the graph in the interval $\left[0, \frac{\pi}{2}\right]$.



Now, we know that $\sin(\pi - x) = \sin x$. So, we may get some other values of the function in the interval $[\pi/2, \pi]$. Moreover, the function is strictly decreasing in this interval. Thus, we may draw it in the interval $[\pi/2, \pi]$. Finally, we draw it in $(\pi, 2\pi)$, using the fact that $\sin(\pi + x) = -\sin x$.

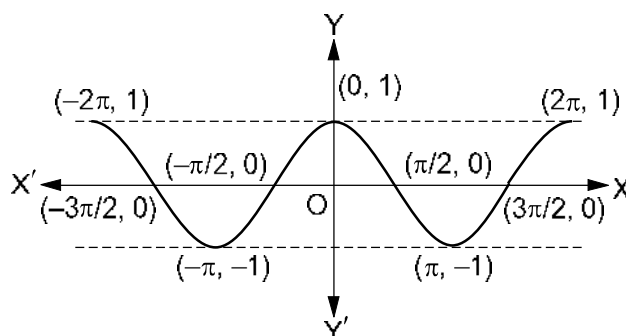
The graph may be completed now by making repetitions over each interval of length 2π .

2. GRAPH OF A COSINE FUNCTION We know that a cosine function is periodic with period 2π . By making use of the table given below, we may first draw it in the interval $[0, \pi/2]$, keeping in view that it is strictly decreasing in this interval.

x	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos x$	1	$\frac{\sqrt{3}}{2} = 0.87$	$\frac{1}{\sqrt{2}} = 0.707$	$\frac{1}{2}$	0

Now, $\cos(\pi - x) = -\cos x$, so, we may draw the graph in the interval $[\pi/2, \pi]$. It is strictly decreasing in this interval also. Further, making use of $\cos(\pi + x) = -\cos x$, we may draw the graph from π to 2π , as shown in the figure.

The graph may now be completed by making repetitions over each interval of length 2π .



3. GRAPH OF A TANGENT FUNCTION We know that a tangent function is a periodic function with period π . Therefore, it is enough to draw the graph over an interval of length π . The complete graph then consists of infinitely many repetitions of the same to the left as well as to the right.

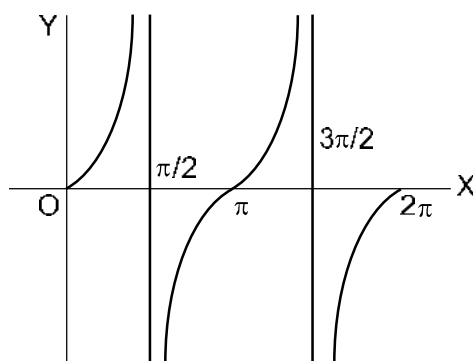
Since $\tan(-x) = -\tan x$, therefore, if $(x, \tan x)$ is any point on the graph then $(-x, -\tan x)$ is also a point on the graph. Thus, we can say that the graph of $y = \tan x$ is symmetrical in opposite quadrants.

Some of the values of the function are given below:

x	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	0	$\frac{\sqrt{3}}{3} = 0.58$	1	$\sqrt{3} = 1.73$	undefined

x	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan x$	$-\sqrt{3} = -1.73$	-1	$-\frac{\sqrt{3}}{3} = -0.58$	0

The graph may thus be drawn as shown below.



Similarly, the graphs of $\operatorname{cosec} x$, $\sec x$ and $\cot x$ may be drawn.

Continuity

CONTINUITY AT A POINT A real function $f(x)$ is said to be continuous at a point a of its domain if $\lim_{x \rightarrow a} f(x)$ exists and equals $f(a)$.

Thus, $f(x)$ is continuous at $x = a$ if

$$\lim_{x \rightarrow a+} f(x) = \lim_{x \rightarrow a-} f(x) = f(a).$$

If $f(x)$ is not continuous at a point, it is said to be *discontinuous* at that point.

REMARK $f(x)$ is discontinuous at $x = a$ in each of the following cases:

- (i) $f(a)$ is not defined
- (ii) $\lim_{x \rightarrow a} f(x)$ does not exist
- (iii) $\lim_{x \rightarrow a} f(x) \neq f(a)$ {removable discontinuity}

SOLVED EXAMPLES

EXAMPLE 1 Show that $f(x) = x^3$ is continuous at $x = 2$.

SOLUTION We have $f(2) = 2^3 = 8$;

$$\lim_{x \rightarrow 2+} f(x) = \lim_{h \rightarrow 0} (2+h)^3 = \lim_{h \rightarrow 0} (8 + h^3 + 12h + 6h^2) = 8;$$

$$\lim_{x \rightarrow 2-} f(x) = \lim_{h \rightarrow 0} (2-h)^3 = \lim_{h \rightarrow 0} (8 - h^3 - 12h + 6h^2) = 8.$$

$$\therefore \lim_{x \rightarrow 2+} f(x) = \lim_{x \rightarrow 2-} f(x) = f(2).$$

Hence, $f(x)$ is continuous at $x = 2$.

EXAMPLE 2 Show that $f(x) = [x]$ is not continuous at $x = n$, where n is an integer.

SOLUTION We have $f(n) = [n] = n$;

$$\lim_{x \rightarrow n+} f(x) = \lim_{h \rightarrow 0} f(n+h) = \lim_{h \rightarrow 0} [n+h] = n \quad \{\because [n+h] = n\};$$

$$\lim_{x \rightarrow n-} f(x) = \lim_{h \rightarrow 0} f(n-h) = \lim_{h \rightarrow 0} [n-h] = (n-1) \quad \{\because [n-h] = (n-1)\}.$$

Thus, $\lim_{x \rightarrow n+} f(x) \neq \lim_{x \rightarrow n-} f(x)$ and therefore, $\lim_{x \rightarrow n} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = n$.

EXAMPLE 3 Show that the function $f(x) = \begin{cases} x, & \text{if } x \text{ is an integer;} \\ 0, & \text{if } x \text{ is not an integer} \end{cases}$

is discontinuous at each integral value of x .

SOLUTION Let $x = n$, where n is an integer. Then, $f(n) = n$;

$$\lim_{x \rightarrow n+} f(x) = \lim_{h \rightarrow 0} f(n+h) = 0$$

[$\because (n+h)$ is not an integer $\Rightarrow f(n+h) = 0$];

and $\lim_{x \rightarrow n-} f(x) = \lim_{h \rightarrow 0} f(n-h) = 0$

[$\because (n-h)$ is not an integer $\Rightarrow f(n-h) = 0$].

$$\therefore \lim_{x \rightarrow n+} f(x) = \lim_{x \rightarrow n-} f(x) = 0.$$

So, $\lim_{x \rightarrow n} f(x) = 0 \neq f(n).$

Hence, $f(x)$ is discontinuous at $x = n$.

EXAMPLE 4 Discuss the continuity of the function $f(x)$ at $x = 0$, if

$$f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x \geq 0. \end{cases} \quad \text{[CBSE 2002]}$$

SOLUTION Clearly, $f(0) = (2 \times 0 + 1) = 1.$

$$\begin{aligned} \lim_{x \rightarrow 0+} f(x) &= \lim_{h \rightarrow 0} f(0+h) \\ &= \lim_{h \rightarrow 0} [2(0+h) + 1] = \lim_{h \rightarrow 0} (2h + 1) = 1. \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0-} f(x) &= \lim_{h \rightarrow 0} f(0-h) \\ &= \lim_{h \rightarrow 0} [2(0-h) - 1] = \lim_{h \rightarrow 0} (-2h - 1) = -1. \end{aligned}$$

Thus, $\lim_{x \rightarrow 0+} f(x) \neq \lim_{x \rightarrow 0-} f(x)$ and therefore, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 5 Show that the function $f(x) = \begin{cases} 3x - 2, & \text{when } x \leq 0 \\ x + 1, & \text{when } x > 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION We have, $f(0) = (3 \times 0 - 2) = -2.$

$$\begin{aligned} \lim_{x \rightarrow 0+} f(x) &= \lim_{h \rightarrow 0} f(0+h) \\ &= \lim_{h \rightarrow 0} (h + 1) = 1. \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0-} f(x) &= \lim_{h \rightarrow 0} f(0-h) \\ &= \lim_{h \rightarrow 0} [3(-h) - 2] = \lim_{h \rightarrow 0} (-3h - 2) = -2. \end{aligned}$$

$\therefore \lim_{x \rightarrow 0+} f(x) \neq \lim_{x \rightarrow 0-} f(x)$ and therefore, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 6 Show that the function $f(x) = \begin{cases} \frac{x}{|x|}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION It is being given that $f(0) = 1$.

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} \frac{h}{|h|} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{-h}{|h|} = \lim_{h \rightarrow 0} \frac{-h}{h} = -1.$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x).$$

So, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 7 Examine the continuity of the function

$$f(x) = \begin{cases} \frac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad \text{at } x = 0. \quad \text{[CBSE 2004]}$$

SOLUTION We have $f(0) = 1$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0+h) \\ &= \lim_{h \rightarrow 0} \frac{|\sin(0+h)|}{(0+h)} = \lim_{h \rightarrow 0} \frac{|\sin h|}{h} = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1. \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0-h) \\ &= \lim_{h \rightarrow 0} \frac{|\sin(-h)|}{-h} = \lim_{h \rightarrow 0} \frac{|-\sin h|}{-h} = \lim_{h \rightarrow 0} \frac{\sin h}{-h} = -1. \end{aligned}$$

$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$. So, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 8 Show that the function $f(x) = 2x - |x|$ is continuous at $x = 0$.

[CBSE 2002]

SOLUTION We have, $f(0) = (2 \times 0) - |0| = 0$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0+h) \\ &= \lim_{h \rightarrow 0} (2h - |h|) = \lim_{h \rightarrow 0} (2h - h) = \lim_{h \rightarrow 0} h = 0. \end{aligned}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0-h)$$

$$= \lim_{h \rightarrow 0} \{2(-h) - |-h|\} = \lim_{h \rightarrow 0} (-2h - h) = \lim_{h \rightarrow 0} (-3h) = 0.$$

Thus, $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 0$ and therefore, $\lim_{x \rightarrow 0} f(x) = 0$.

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0) = 0.$$

Hence, $f(x)$ is continuous at $x = 0$.

EXAMPLE 9 Let $f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3. \end{cases}$

Show that $f(x)$ is continuous at $x = -3$ but discontinuous at $x = 3$.

SOLUTION We have $f(-3) = |-3| + 3 = (3 + 3) = 6$.

$$\begin{aligned} \lim_{x \rightarrow (-3)^+} f(x) &= \lim_{h \rightarrow 0} f(-3 + h) \\ &= \lim_{h \rightarrow 0} \{-2(-3 + h)\} = \lim_{h \rightarrow 0} (6 - 2h) = 6. \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow (-3)^-} f(x) &= \lim_{h \rightarrow 0} f(-3 - h) \\ &= \lim_{h \rightarrow 0} (|-3 - h| + 3) = \lim_{h \rightarrow 0} \{|-(3 + h)| + 3\} \\ &= \lim_{h \rightarrow 0} \{(3 + h) + 3\} = 6. \end{aligned}$$

$$\therefore \lim_{x \rightarrow (-3)^+} f(x) = \lim_{x \rightarrow (-3)^-} f(x) = 6. \quad \text{So, } \lim_{x \rightarrow (-3)} f(x) = 6.$$

$$\text{Thus, } \lim_{x \rightarrow (-3)} f(x) = f(-3) = 6.$$

Hence, $f(x)$ is continuous at $x = -3$.

$$\text{Now, } f(3) = (6 \times 3 + 2) = 20.$$

$$\begin{aligned} \lim_{x \rightarrow 3^-} f(x) &= \lim_{h \rightarrow 0} f(3 - h) \\ &= \lim_{h \rightarrow 0} \{-2(3 - h)\} = \lim_{h \rightarrow 0} (-6 + 2h) = -6. \end{aligned}$$

$$\text{Thus, } f(3) \neq \lim_{x \rightarrow 3^-} f(x).$$

Hence, $f(x)$ is discontinuous at $x = 3$.

EXAMPLE 10 Find the value of k for which

$$f(x) = \begin{cases} kx + 5, & \text{when } x \leq 2 \\ x - 1, & \text{when } x > 2 \end{cases}$$

is continuous at $x = 2$.

[CBSE 2002]

SOLUTION We have, $f(2) = (k \times 2 + 5) = (2k + 5)$.

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2 + h)$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \{(2+h) - 1\} = \lim_{h \rightarrow 0} (1+h) = 1. \\
\lim_{x \rightarrow 2^-} f(x) &= \lim_{h \rightarrow 0} f(2-h) \\
&= \lim_{h \rightarrow 0} \{k(2-h) + 5\} = \lim_{h \rightarrow 0} \{(2k+5) - kh\} = (2k+5).
\end{aligned}$$

Now, $\lim_{x \rightarrow 2} f(x)$ exists only when $2k+5 = 1$, i.e., when $k = -2$.

When $k = -2$, we have $\lim_{x \rightarrow 2} f(x) = f(2) = 1$.

Hence, $f(x)$ is continuous at $x = 2$ when $k = -2$.

EXAMPLE 11 If the following function $f(x)$ is continuous at $x = 0$, find the value of k :

$$f(x) = \begin{cases} \frac{1 - \cos 2x}{2x^2}, & x \neq 0 \\ k, & x = 0. \end{cases} \quad \text{[CBSE 2008]}$$

SOLUTION We have, $f(0) = k$.

$$\begin{aligned}
\lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0+h) \\
&= \lim_{h \rightarrow 0} \frac{(1 - \cos 2h)}{2h^2} = \lim_{h \rightarrow 0} \frac{2\sin^2 h}{2h^2} = \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right)^2 \\
&= \left\{ \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \right\}^2 = (1)^2 = 1.
\end{aligned}$$

$$\begin{aligned}
\lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0-h) \\
&= \lim_{h \rightarrow 0} \left\{ \frac{1 - \cos 2(-h)}{2(-h)^2} \right\} = \lim_{h \rightarrow 0} \frac{\{1 - \cos(-2h)\}}{2h^2} \\
&= \lim_{h \rightarrow 0} \frac{(1 - \cos 2h)}{2h^2} = \lim_{h \rightarrow 0} \frac{2\sin^2 h}{2h^2} = \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right)^2 \\
&= \left\{ \lim_{h \rightarrow 0} \frac{\sin h}{h} \right\}^2 = 1^2 = 1.
\end{aligned}$$

Since $f(x)$ is continuous at $x = 0$, we must have

$$f(0) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) \Rightarrow k = 1.$$

EXAMPLE 12 For what value of k is the following function continuous at $x = 0$?

$$f(x) = \begin{cases} \frac{1 - \cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases} \quad \text{[CBSE 2014C]}$$

SOLUTION We have, $f(0) = k$.

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0+h) \\
 &= \lim_{h \rightarrow 0} \frac{(1 - \cos 4h)}{8h^2} = \lim_{h \rightarrow 0} \frac{2\sin^2 2h}{8h^2} = \lim_{h \rightarrow 0} \left(\frac{\sin 2h}{2h} \right)^2 \\
 &= 1^2 = 1.
 \end{aligned}$$

$$\begin{aligned}
 \lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0-h) \\
 &= \lim_{h \rightarrow 0} \frac{\{1 - \cos 4(-h)\}}{8(-h)^2} = \lim_{h \rightarrow 0} \frac{\{1 - \cos(-4h)\}}{8h^2} = \lim_{h \rightarrow 0} \frac{(1 - \cos 4h)}{8h^2} \\
 &\quad [\because \cos(-\theta) = \cos \theta] \\
 &= \lim_{h \rightarrow 0} \frac{2\sin^2 2h}{8h^2} = \lim_{h \rightarrow 0} \left(\frac{\sin 2h}{2h} \right)^2 \\
 &= 1^2 = 1.
 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 0} f(x) = 1.$$

For continuity at $x = 0$, we must have

$$f(0) = \lim_{x \rightarrow 0} f(x) \Rightarrow k = 1 \quad \left[\because f(0) = k \text{ and } \lim_{x \rightarrow 0} f(x) = 1 \right].$$

EXAMPLE 13 Given that

$$f(x) = \begin{cases} \frac{(1 - \cos 4x)}{x^2}, & \text{if } x < 0 \\ a, & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & \text{if } x > 0. \end{cases}$$

If $f(x)$ is continuous at $x = 0$, find the value of a . [CBSE 2006C, '12C, '13C]

SOLUTION We have, $f(0) = a$.

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0+h) \\
 &= \lim_{h \rightarrow 0} \left\{ \frac{\sqrt{h}}{\sqrt{16 + \sqrt{h}} - 4} \right\} \\
 &= \lim_{h \rightarrow 0} \left\{ \frac{\sqrt{h}}{\sqrt{16 + \sqrt{h}} - 4} \times \frac{\sqrt{16 + \sqrt{h}} + 4}{\sqrt{16 + \sqrt{h}} + 4} \right\} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{h} \cdot \{\sqrt{16 + \sqrt{h}} + 4\}}{\{(16 + \sqrt{h}) - 16\}} = \lim_{h \rightarrow 0} \frac{\sqrt{h} \cdot \{\sqrt{16 + \sqrt{h}} + 4\}}{\sqrt{h}} \\
 &= \lim_{h \rightarrow 0} \{\sqrt{16 + \sqrt{h}} + 4\} = (4 + 4) = 8.
 \end{aligned}$$

$$\begin{aligned}
\lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h) \\
&= \lim_{h \rightarrow 0} \frac{\{1 - \cos 4(-h)\}}{(-h)^2} = \lim_{h \rightarrow 0} \frac{\{1 - \cos(-4h)\}}{h^2} \\
&= \lim_{h \rightarrow 0} \frac{(1 - \cos 4h)}{h^2} = \lim_{h \rightarrow 0} \frac{2\sin^2 2h}{h^2} \\
&= (2 \times 4) \cdot \lim_{h \rightarrow 0} \frac{(\sin^2 2h)}{(2h)^2} = 8 \cdot \lim_{h \rightarrow 0} \left(\frac{\sin 2h}{2h} \right)^2 \\
&= (8 \times 1^2) = 8.
\end{aligned}$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 8 \Rightarrow \lim_{x \rightarrow 0} f(x) = 8.$$

But, by continuity of f at $x = 0$, we have

$$\lim_{x \rightarrow 0} f(x) = f(0) \Rightarrow a = 8 \quad [\because f(0) = a \text{ and } \lim_{x \rightarrow 0} f(x) = 8].$$

Hence, $a = 8$.

EXAMPLE 14 If the following function $f(x)$ is continuous at $x = 0$, find the values of a , b and c .

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin x}{x}, & \text{if } x < 0 \\ c, & \text{if } x = 0 \\ \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}}, & \text{if } x > 0. \end{cases}$$

[CBSE 2008]

SOLUTION We have, $f(0) = c$.

$$\begin{aligned}
\lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0-h) \\
&= \lim_{h \rightarrow 0} \frac{\sin(a+1)(-h) + \sin(-h)}{(-h)} = \lim_{h \rightarrow 0} \frac{-\{\sin(a+1)h + \sin h\}}{-h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(a+1)h + \sin h}{h} = \lim_{h \rightarrow 0} \frac{2\sin\left(\frac{a}{2}+1\right)h \cdot \cos \frac{ah}{2}}{h} \\
&= 2 \cdot \lim_{h \rightarrow 0} \left\{ \frac{\sin\left(\frac{a}{2}+1\right)h}{\left(\frac{a}{2}+1\right)h} \cdot \left(\frac{a}{2}+1\right) \cdot \cos \frac{ah}{2} \right\}
\end{aligned}$$

$$= 2\left(\frac{a}{2} + 1\right) \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\left(\frac{a}{2} + 1\right)h\right)}{\left(\frac{a}{2} + 1\right)h} \cdot \lim_{h \rightarrow 0} \cos \frac{ah}{2}$$

$$= (a + 2) \times 1 \times 1 = (a + 2).$$

$$\begin{aligned} \lim_{x \rightarrow 0+} f(x) &= \lim_{h \rightarrow 0} f(0 + h) \\ &= \lim_{h \rightarrow 0} \left\{ \frac{[\sqrt{h + bh^2} - \sqrt{h}]}{bh^{3/2}} \times \frac{[\sqrt{h + bh^2} + \sqrt{h}]}{[\sqrt{h + bh^2} + \sqrt{h}]} \right\} \\ &= \lim_{h \rightarrow 0} \frac{(h + bh^2 - h)}{bh^{3/2}(\sqrt{h + bh^2} + \sqrt{h})} = \lim_{h \rightarrow 0} \frac{bh^2}{bh^2(\sqrt{1 + bh} + 1)} \\ &= \lim_{h \rightarrow 0} \frac{1}{(\sqrt{1 + bh} + 1)} = \frac{1}{2}. \end{aligned}$$

Since $f(x)$ is continuous at $x = 0$, we have

$$f(0) = \lim_{x \rightarrow 0-} f(x) = \lim_{x \rightarrow 0+} f(x) \Rightarrow c = \frac{1}{2} \text{ and } a + 2 = \frac{1}{2}.$$

$$\therefore c = \frac{1}{2} \text{ and } a = \frac{-3}{2}.$$

EXAMPLE 15 If the function $f(x) = \begin{cases} 3ax + b, & \text{for } x > 1 \\ 11, & \text{for } x = 1 \\ 5ax - 2b, & \text{for } x < 1 \end{cases}$

is continuous at $x = 1$, find the values of a and b .

[CBSE 2012C]

SOLUTION We have, $f(1) = 11$.

$$\begin{aligned} \lim_{x \rightarrow 1+} f(x) &= \lim_{h \rightarrow 0} f(1 + h) \\ &= \lim_{h \rightarrow 0} \{3a(1 + h) + b\} = \lim_{h \rightarrow 0} \{(3a + b) + 3ah\} \\ &= (3a + b). \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 1-} f(x) &= \lim_{h \rightarrow 0} f(1 - h) \\ &= \lim_{h \rightarrow 0} \{5a(1 - h) - 2b\} = \lim_{h \rightarrow 0} \{(5a - 2b) - 5ah\} \\ &= (5a - 2b). \end{aligned}$$

Since $f(x)$ is continuous at $x = 1$, we have

$$\lim_{x \rightarrow 1+} f(x) = \lim_{x \rightarrow 1-} f(x) = f(1).$$

$$\therefore 3a + b = 5a - 2b = 11.$$

On solving $(3a + b = 11)$ and $(5a - 2b = 11)$, we get $a = 3$, $b = 2$.

Hence, $a = 3$, $b = 2$.

EXAMPLE 16 For what value of k is the function

$$f(x) = \begin{cases} k(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

(i) continuous at $x = 0$?

(ii) continuous at $x = 1$?

(iii) continuous at $x = -1$?

SOLUTION

(i) We have $f(0) = k(0 - 2 \times 0) = 0$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0 + h) \\ &= \lim_{h \rightarrow 0} \{4(0 + h) + 1\} = \lim_{h \rightarrow 0} (4h + 1) = 1. \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0 - h) \\ &= \lim_{h \rightarrow 0} k\{(-h)^2 - 2(-h)\} = \lim_{h \rightarrow 0} k(h^2 + 2h) = 0. \end{aligned}$$

Thus, $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$, and thus $\lim_{x \rightarrow 0} f(x)$ does not exist.

So, $f(x)$ is not continuous at $x = 0$ for any value of k .

(ii) $f(1) = (4 \times 1 + 1) = 5$.

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{h \rightarrow 0} f(1 + h) \\ &= \lim_{h \rightarrow 0} \{4(1 + h) + 1\} = \lim_{h \rightarrow 0} (5 + 4h) = 5. \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{h \rightarrow 0} f(1 - h) \\ &= \lim_{h \rightarrow 0} \{4(1 - h) + 1\} = \lim_{h \rightarrow 0} (5 - 4h) = 5. \end{aligned}$$

Thus, $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(x) = 5$.

Hence, $f(x)$ is continuous at $x = 1$ for every real value of k .

(iii) $f(-1) = k\{(-1)^2 - 2 \times (-1)\} = 3k$.

$$\begin{aligned} \lim_{x \rightarrow (-1)^+} f(x) &= \lim_{h \rightarrow 0} f(-1 + h) \\ &= \lim_{h \rightarrow 0} k\{(-1 + h)^2 - 2(-1 + h)\} \\ &= \lim_{h \rightarrow 0} k\{1 + h^2 - 2h + 2 - 2h\} \\ &= \lim_{h \rightarrow 0} k(3 + h^2 - 4h) = 3k. \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow (-1)^-} f(x) &= \lim_{h \rightarrow 0} f(-1 - h) \\ &= \lim_{h \rightarrow 0} k\{(-1 - h)^2 - 2(-1 - h)\} = \lim_{h \rightarrow 0} k\{1 + h^2 + 2h + 2 + 2h\} \\ &= \lim_{h \rightarrow 0} k(3 + 4h + h^2) = 3k. \end{aligned}$$

$$\text{Thus, } \lim_{x \rightarrow (-1)^+} f(x) = \lim_{x \rightarrow (-1)^-} f(x) = f(-1) = 3k.$$

Hence, $f(x)$ is continuous at $x = -1$ for each real value of k .

EXAMPLE 17 Show that the function $f(x) = \begin{cases} \frac{\sin^2 ax}{x^2}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

Redefine the function in such a way that it becomes continuous at $x = a$.

SOLUTION Clearly, $f(0) = 1$.

$$\text{Also, } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin^2 ax}{x^2} = a^2 \cdot \lim_{ax \rightarrow 0} \left(\frac{\sin ax}{ax} \right)^2 = a^2.$$

Since $\lim_{x \rightarrow 0} f(x) \neq f(0)$, $f(x)$ is discontinuous at $x = 0$.

However, it becomes continuous if $f(0) = a^2$.

$$\text{So, the desired function is } f(x) = \begin{cases} \frac{\sin^2 ax}{x^2}, & \text{when } x \neq 0 \\ a^2, & \text{when } x = 0. \end{cases}$$

EXAMPLE 18 Is the function $f(x) = \frac{(3x + 4 \tan x)}{x}$ continuous at $x = 0$? If not, how may the function be defined to make it continuous at this point?

SOLUTION Since $f(x)$ is not defined at $x = 0$, it cannot be continuous at $x = 0$.

$$\begin{aligned} \text{However, } \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \left(\frac{3x + 4 \tan x}{x} \right) = \lim_{x \rightarrow 0} \left[3 + 4 \cdot \frac{\sin x}{x} \cdot \frac{1}{\cos x} \right] \\ &= 3 + 4 \cdot \lim_{x \rightarrow 0} \left\{ \frac{\sin x}{x} \right\} \cdot \left\{ \lim_{x \rightarrow 0} \frac{1}{\cos x} \right\} = 7. \end{aligned}$$

So, in order to make $f(x)$ continuous at $x = 0$, we define it as

$$f(x) = \begin{cases} \frac{(3x + 4 \tan x)}{x}, & \text{when } x \neq 0 \\ 7, & \text{when } x = 0. \end{cases}$$

EXAMPLE 19 Show that the function $f(x) = \begin{cases} \left(\frac{e^{1/x} - 1}{e^{1/x} + 1} \right), & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION Clearly, $f(0) = 0$.

$$\begin{aligned}\text{Now, } \lim_{x \rightarrow 0} f(x) &= \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} \left(\frac{e^{1/h} - 1}{e^{1/h} + 1} \right) \\ &= \lim_{h \rightarrow 0} \frac{e^{1/h} \left(1 - \frac{1}{e^{1/h}} \right)}{e^{1/h} \left(1 + \frac{1}{e^{1/h}} \right)} = \lim_{h \rightarrow 0} \frac{\left(1 - \frac{1}{e^{1/h}} \right)}{\left(1 + \frac{1}{e^{1/h}} \right)} = 1.\end{aligned}$$

$$\begin{aligned}\text{And, } \lim_{x \rightarrow 0} f(x) &= \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} \left(\frac{e^{-1/h} - 1}{e^{-1/h} + 1} \right) \\ &= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{e^{1/h}} - 1 \right)}{\left(\frac{1}{e^{1/h}} + 1 \right)} = -1.\end{aligned}$$

Thus, $\lim_{x \rightarrow 0+} f(x) \neq \lim_{x \rightarrow 0-} f(x)$, and therefore, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXERCISE 9A

1. Show that $f(x) = x^2$ is continuous at $x = 2$.
2. Show that $f(x) = (x^2 + 3x + 4)$ is continuous at $x = 1$.

Prove that

$$3. f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3}, & \text{when } x \neq 3; \\ 5, & \text{when } x = 3 \end{cases} \text{ is continuous at } x = 3.$$

$$4. f(x) = \begin{cases} \frac{x^2 - 25}{x - 5}, & \text{when } x \neq 5; \\ 10, & \text{when } x = 5 \end{cases} \text{ is continuous at } x = 5.$$

$$5. f(x) = \begin{cases} \frac{\sin 3x}{x}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases} \text{ is discontinuous at } x = 0.$$

$$6. f(x) = \begin{cases} \frac{1 - \cos x}{x^2}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases} \text{ is discontinuous at } x = 0. \quad [\text{CBSE 2000}]$$

$$7. f(x) = \begin{cases} 2 - x, & \text{when } x < 2; \\ 2 + x, & \text{when } x \geq 2 \end{cases} \text{ is discontinuous at } x = 2.$$

$$8. f(x) = \begin{cases} 3 - x, & \text{when } x \leq 0; \\ x^2, & \text{when } x > 0 \end{cases} \text{ is discontinuous at } x = 0.$$

9. $f(x) = \begin{cases} 5x - 4, & \text{when } 0 < x \leq 1; \\ 4x^2 - 3x, & \text{when } 1 < x < 2 \end{cases}$ is continuous at $x = 1$. [CBSE 2001C]
10. $f(x) = \begin{cases} x - 1, & \text{when } 1 \leq x < 2; \\ 2x - 3, & \text{when } 2 \leq x \leq 3 \end{cases}$ is continuous at $x = 2$.
11. $f(x) = \begin{cases} \cos x, & \text{when } x \geq 0; \\ -\cos x, & \text{when } x < 0 \end{cases}$ is discontinuous at $x = 0$.
12. $f(x) = \begin{cases} \frac{|x-a|}{x-a}, & \text{when } x \neq a; \\ 1, & \text{when } x = a \end{cases}$ is discontinuous at $x = a$.
13. $f(x) = \begin{cases} \frac{1}{2}(x - |x|), & \text{when } x \neq 0; \\ 2, & \text{when } x = 0 \end{cases}$ is discontinuous at $x = 0$.
14. $f(x) = \begin{cases} \sin \frac{1}{x}, & \text{when } x \neq 0; \\ 0, & \text{when } x = 0 \end{cases}$ is discontinuous at $x = 0$.
15. $f(x) = \begin{cases} 2x, & \text{when } x < 2; \\ 2, & \text{when } x = 2; \\ x^2, & \text{when } x > 2 \end{cases}$ is discontinuous at $x = 2$.
16. $f(x) = \begin{cases} -x, & \text{when } x < 0; \\ 1, & \text{when } x = 0; \\ x, & \text{when } x > 0 \end{cases}$ is discontinuous at $x = 0$.
17. Find the value of k for which
- $$f(x) = \begin{cases} \frac{\sin 2x}{5x}, & \text{when } x \neq 0; \\ k, & \text{when } x = 0 \end{cases} \text{ is continuous at } x = 0.$$
18. Find the value of λ for which
- $$f(x) = \begin{cases} \frac{x^2 - 2x - 3}{x + 1}, & \text{when } x \neq -1; \\ \lambda, & \text{when } x = -1 \end{cases} \text{ is continuous at } x = -1.$$
19. For what value of k is the following function continuous at $x = 2$?
- $$f(x) = \begin{cases} 2x + 1, & x < 2 \\ k, & x = 2 \\ 3x - 1, & x > 2 \end{cases} \quad \text{[CBSE 2008]}$$
20. For what value of k is the function
- $$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & \text{when } x \neq 3; \\ k, & \text{when } x = 3 \end{cases} \text{ is continuous at } x = 3?$$

21. Find the value of k for which the function

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2}; \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \text{ is continuous at } x = \frac{\pi}{2}. \quad [\text{CBSE 2012C}]$$

22. Show that the function: $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

23. Show that: $f(x) = \begin{cases} x+1, & \text{if } x \geq 1; \\ x^2+1, & \text{if } x < 1 \end{cases}$ is continuous at $x = 1$. [CBSE 2007]

24. Show that: $f(x) = \begin{cases} x^3-3, & \text{if } x \leq 2; \\ x^2+1, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$. [CBSE 2007]

25. Find the values of a and b such that the following function is continuous:

$$f(x) = \begin{cases} 5, & \text{when } x \leq 2 \\ ax+b, & \text{when } 2 < x < 10 \\ 21, & \text{when } x \geq 10. \end{cases} \quad [\text{CBSE 2011}]$$

26. Find the values of a for which the function f , defined as

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \leq 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases} \text{ is continuous at } x = 0. \quad [\text{CBSE 2011}]$$

27. Prove that the function f given by $f(x) = |x-3|$, $x \in \mathbb{R}$ is continuous but not differentiable at $x = 3$. [CBSE 2012C]

ANSWERS (EXERCISE 9A)

17. $k = \frac{2}{5}$ 18. $\lambda = -4$ 19. $k = 5$ 20. $k = 6$ 21. $k = 6$ 25. $a = 2, b = 1$ 26. $a = \frac{1}{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 9A)

12. $\lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} \frac{|a+h-a|}{(a+h-a)} = \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$

$\lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} \frac{|a-h-a|}{(a-h-a)} = \lim_{h \rightarrow 0} \frac{|-h|}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = -1.$

13. $\lim_{x \rightarrow 0} f(x) = \lim_{h \rightarrow 0} \frac{1}{2}(h - |h|) = 0$

and $\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} \frac{1}{2}(-h - |-h|) = -\lim_{h \rightarrow 0} \frac{h}{2} = 0.$

14. $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist, since as $x \rightarrow 0$, the value of $\sin(1/x)$ oscillates between -1 and 1 , and it does not approach a definite number.

$$21. f\left(\frac{\pi}{2} - 0\right) = \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \rightarrow 0} \frac{k \sin h}{2h} = \frac{k}{2} \lim_{h \rightarrow 0} \frac{\sin h}{h} = \left(\frac{k}{2} \times 1\right) = \frac{k}{2}.$$

$$f\left(\frac{\pi}{2} + 0\right) = \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h \rightarrow 0} \frac{-k \sin h}{-2h} = \frac{k}{2} \lim_{h \rightarrow 0} \frac{\sin h}{h} = \left(\frac{k}{2} \times 1\right) = \frac{k}{2}.$$

$$\therefore \frac{k}{2} = 3 \Rightarrow k = 6.$$

$$22. f(0 - 0) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} (-h)^2 \sin\left(\frac{-1}{h}\right) = \lim_{h \rightarrow 0} -h^2 \sin \frac{1}{h} = (0 \times \text{a finite quantity}) = 0.$$

$$f(0 + 0) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} h^2 \sin \frac{1}{h} = (0 \times \text{a finite quantity}) = 0.$$

$$\therefore f(0 - 0) = f(0 + 0) = f(0).$$

Hence, $f(x)$ is continuous at $x = 0$.

Continuous Functions

CONTINUITY IN AN INTERVAL A function $f(x)$ is said to be continuous in an open interval $]a, b[$ if it is continuous at each point of $]a, b[$.

If $f(x)$ is defined on a closed interval $[a, b]$, we say that

(i) f is continuous at a if $\lim_{x \rightarrow a^+} f(x) = f(a)$;

(ii) f is continuous at b if $\lim_{x \rightarrow b^-} f(x) = f(b)$;

(iii) f is continuous in $[a, b]$ if it is continuous at a , at b and at each point of $]a, b[$.

CONTINUOUS FUNCTIONS A function $f(x)$ is said to be continuous if it is continuous at each point of its domain.

Algebra of Continuous Functions

THEOREM 1 Every constant function is continuous.

PROOF Let $f(x) = c$, where c is constant.

Clearly, the domain of a constant function is R .

Let a be an arbitrary real number.

Then, $f(a) = c$ and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} c = c$ [$\because f(x) = c$].

$\therefore \lim_{x \rightarrow a} f(x) = f(a)$.

Thus, $f(x)$ is continuous at $x = a$ for all $a \in R$.

Hence, $f(x)$ is continuous.

THEOREM 2 *Show that the identity function is continuous.*

PROOF Let $f(x) = x$ for all $x \in \mathbb{R}$.

Let a be an arbitrary real number. Then, $f(a) = a$.

And, $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} x = a \quad [\because f(x) = x]$.

$\therefore \lim_{x \rightarrow a} f(x) = f(a) = a$.

This shows that $f(x)$ is continuous at $x = a$ for all $a \in \mathbb{R}$.

Hence, the identity function is continuous.

THEOREM 3 *If f and g be continuous functions then*

- (i) $f + g$ is continuous (ii) $f - g$ is continuous
(iii) cf is continuous (iv) fg is continuous

(v) $\left(\frac{f}{g}\right)$ is continuous at those points where $g(x) \neq 0$

PROOF (i) Let $\text{dom}(f) = D_1$ and $\text{dom}(g) = D_2$. Then, $\text{dom}(f + g) = D_1 \cap D_2$.

Let $a \in D_1 \cap D_2$. Then, $a \in D_1$ and $a \in D_2$.

By continuity of f and g , we have

$$\lim_{x \rightarrow a} f(x) = f(a) \text{ and } \lim_{x \rightarrow a} g(x) = g(a).$$

$$\begin{aligned} \therefore \lim_{x \rightarrow a} (f + g)(x) &= \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) \\ &= f(a) + g(a) = (f + g)(a). \end{aligned}$$

This shows that $(f + g)$ is continuous at a for all $a \in D_1 \cap D_2$.

Hence, $(f + g)$ is continuous.

Similarly, (ii) may be proved.

(iii) Let $\text{dom}(f) = D_1$. Then, $\text{dom}(cf) = D_1$. Let $a \in D_1$.

Then, by the continuity of f , we have $\lim_{x \rightarrow a} f(x) = f(a)$.

$$\therefore \lim_{x \rightarrow a} (cf)(x) = \lim_{x \rightarrow a} c \cdot f(x) = c \cdot \lim_{x \rightarrow a} f(x) = c \cdot f(a) = (cf)(a).$$

This shows that (cf) is continuous at a for all $a \in D_1$.

Hence, cf is continuous.

(iv) Let $\text{dom}(f) = D_1$ and $\text{dom}(g) = D_2$. Then, $\text{dom}(fg) = D_1 \cap D_2$.

Let $a \in D_1 \cap D_2$. Then, $a \in D_1$ and $a \in D_2$.

By continuity of f and g , we have

$$\lim_{x \rightarrow a} f(x) = f(a) \text{ and } \lim_{x \rightarrow a} g(x) = g(a).$$

$$\begin{aligned} \therefore \lim_{x \rightarrow a} (fg)(x) &= \lim_{x \rightarrow a} [f(x) \cdot g(x)] \\ &= \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = f(a) \cdot g(a) = (fg)(a). \end{aligned}$$

This shows that fg is continuous.

(v) Let $\text{dom}(f) = D_1$ and $\text{dom}(g) = D_2$.

Then, $\text{dom}\left(\frac{f}{g}\right) = [(D_1 \cap D_2) - \{x : g(x) = 0\}] = D$ (say).

Let $a \in D$. Then $a \in D_1$, $a \in D_2$ and $g(a) \neq 0$.

By continuity of f and g , we have

$$\lim_{x \rightarrow a} f(x) = f(a) \text{ and } \lim_{x \rightarrow a} g(x) = g(a).$$

$$\therefore \lim_{x \rightarrow a} \left(\frac{f}{g}\right)(x) = \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a), \text{ since } g(a) \neq 0.$$

Thus, $\left(\frac{f}{g}\right)$ is continuous at a for all $a \in D$. Hence, $\left(\frac{f}{g}\right)$ is continuous.

THEOREM 4 Every polynomial function is continuous.

PROOF Let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ be a polynomial. We shall prove the theorem by induction on n .

When $n = 0$ then $f(x) = a_0$, which being a constant function is continuous.

When $n = 1$ then $f(x) = a_0 + a_1x$.

Clearly, $f(x)$ is the sum of a constant function and a multiple of the identity function. It, being the sum of two continuous functions, is continuous.

Let every polynomial of degree at most n be continuous.

Consider a general polynomial of degree $(n + 1)$, namely,

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n + a_{n+1}x^{n+1}.$$

This can be written as $a_0 + x(a_1 + a_2x + \dots + a_nx^{n-1} + a_{n+1}x^n)$.

This is the sum of a constant function a_0 (which is continuous) and the product of the identity function x (which is continuous) and the polynomial function $a_1 + a_2x + \dots + a_{n+1}x^n$ of degree at most n (which we assumed to be continuous).

Therefore, it is continuous.

Thus, the continuity of a polynomial of degree n implies the continuity of a polynomial of degree $(n + 1)$.

Hence, by the principle of induction, every polynomial function is continuous.

THEOREM 5 Every rational function is continuous.

PROOF Let $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Then, $\text{dom}[p(x)] = R$ and $\text{dom}[q(x)] = R$.

$$\therefore \text{dom}[f(x)] = \text{dom}\left\{\frac{p(x)}{q(x)}\right\} = R \cap R - \{x : q(x) = 0\} = R - \{x : q(x) = 0\}.$$

Let a be an arbitrary element of $\text{dom } f(x)$.

Then, $a \in \mathbb{R}$ and $q(a) \neq 0$.

But, every polynomial function being continuous, we have

$$\lim_{x \rightarrow a} p(x) = p(a) \text{ and } \lim_{x \rightarrow a} q(x) = q(a).$$

$$\therefore \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \frac{\lim_{x \rightarrow a} p(x)}{\lim_{x \rightarrow a} q(x)} = \frac{p(a)}{q(a)}, \text{ where } q(a) \neq 0.$$

This shows that $f(x)$ is continuous at $x = a$ for all $a \in \text{dom } f(x)$.

Hence, every rational function is continuous.

THEOREM 6 Let f and g be real functions such that $f \circ g$ is defined.

If g is continuous at a and f is continuous at $g(a)$, show that $f \circ g$ is continuous at a .

PROOF Since $f \circ g$ is defined, we have $\text{range}(g) \subseteq \text{dom}(f)$.

Since g is continuous at a , we have $\lim_{x \rightarrow a} g(x) = g(a)$ (i)

Also, f being continuous at $g(a)$, we have $\lim_{y \rightarrow g(a)} f(y) = f\{g(a)\}$ (ii)

$$\begin{aligned} \therefore \lim_{x \rightarrow a} (f \circ g)(x) &= \lim_{x \rightarrow a} f\{g(x)\} = \lim_{g(x) \rightarrow g(a)} f\{g(x)\} \\ & \quad [\because x \rightarrow a \Rightarrow g(x) \rightarrow g(a) \text{ from (i)}] \\ &= \lim_{y \rightarrow g(a)} f(y) = f\{g(a)\} \quad [\text{using (ii)}] \\ &= (f \circ g)(a). \end{aligned}$$

Thus, $(f \circ g)$ is continuous at a .

THEOREM 7 The composite of two continuous functions is continuous.

PROOF Let f and g be continuous functions such that $g \circ f$ is defined.

Then, $\text{range}(f) \subseteq \text{dom}(g)$. Let $a \in \text{dom}(f)$.

Then, $a \in \text{dom}(f) \Rightarrow f(a) \in \text{range}(f)$

$$\Rightarrow f(a) \in \text{dom}(g) \quad [\because \text{range}(f) \subseteq \text{dom}(g)].$$

Thus, f is continuous at a , and g is continuous at $f(a)$.

Consequently, $\lim_{x \rightarrow a} f(x) = f(a)$ (i)

And, $\lim_{y \rightarrow f(a)} g(y) = g[f(a)]$ (ii)

$$\begin{aligned} \therefore \lim_{x \rightarrow a} (g \circ f)(x) &= \lim_{x \rightarrow a} g[f(x)] \\ &= \lim_{f(x) \rightarrow f(a)} g[f(x)] \quad [\because x \rightarrow a \Rightarrow f(x) \rightarrow f(a) \text{ from (i)}] \\ &= \lim_{y \rightarrow f(a)} g(y) = g[f(a)] \quad [\text{using (ii)}] \\ &= (g \circ f)(a). \end{aligned}$$

This shows that $g \circ f$ is continuous at a for all $a \in \text{dom}(f)$.

Hence $g \circ f$ is continuous.

An important result

For continuity of f at a , it is sufficient to show that $\lim_{h \rightarrow 0} f(a+h) = f(a)$, since f is continuous $\Leftrightarrow \lim_{x \rightarrow a} f(x) = f(a)$

$$\begin{aligned} &\Leftrightarrow \lim_{(a+h) \rightarrow a} f(a+h) = f(a) \quad [\text{putting } x = (a+h)] \\ &\Leftrightarrow \lim_{h \rightarrow 0} f(a+h) = f(a). \end{aligned}$$

THEOREM 8 The exponential function is continuous.

PROOF Let $f(x) = e^x$. Then, clearly, $\text{dom}(f) = \mathbb{R}$.

Let a be an arbitrary real number. Then,

$$\lim_{x \rightarrow a} e^x = \lim_{h \rightarrow 0} e^{a+h} = \lim_{h \rightarrow 0} e^a \cdot e^h = e^a \cdot \lim_{h \rightarrow 0} e^h = e^a \times 1 = e^a.$$

Thus, $f(x) = e^x$ is continuous at $x = a$ for all $a \in \mathbb{R}$.

Hence, the exponential function is continuous.

THEOREM 9 (i) The sine function is continuous.

(ii) The cosine function is continuous.

(iii) The tangent function is continuous.

PROOF (i) Let $f(x) = \sin x$. Then, clearly $\text{dom}(f) = \mathbb{R}$.

Let a be an arbitrary real number. Then,

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} \sin x = \lim_{h \rightarrow 0} \sin(a+h) = \lim_{h \rightarrow 0} [\sin a \cos h + \cos a \sin h] \\ &= \sin a \cdot \lim_{h \rightarrow 0} \cos h + \cos a \cdot \lim_{h \rightarrow 0} \sin h \\ &= (\sin a \times 1 + \cos a \times 0) = \sin a = f(a). \end{aligned}$$

Thus, $\lim_{x \rightarrow a} f(x) = f(a)$ for all $a \in \mathbb{R}$.

$\therefore f(x) = \sin x$ is continuous at a for all $a \in \mathbb{R}$.

Hence, $\sin x$ is continuous.

(ii) Let $f(x) = \cos x$. Clearly, $\text{dom}(f) = \mathbb{R}$.

Let a be an arbitrary real number. Then,

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} \cos x = \lim_{h \rightarrow 0} \cos(a+h) = \lim_{h \rightarrow 0} [\cos a \cos h - \sin a \sin h] \\ &= \cos a \cdot \lim_{h \rightarrow 0} \cos h - \sin a \cdot \lim_{h \rightarrow 0} \sin h = (\cos a \times 1 - \sin a \times 0) \\ &= \cos a = f(a). \end{aligned}$$

Thus, $\lim_{x \rightarrow a} f(x) = f(a)$ for all $a \in \mathbb{R}$.

$\therefore f(x) = \cos x$ is continuous at a for all $a \in \mathbb{R}$.

Hence, $\cos x$ is continuous.

(iii) We have $\tan x = \frac{\sin x}{\cos x}$ and $\text{dom}(\tan x) = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{I}\}$.

Let $a \in \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{I}\}$. Then,

$$\begin{aligned}
\lim_{x \rightarrow a} \tan x &= \lim_{x \rightarrow a} \frac{\sin x}{\cos x} = \frac{\lim_{h \rightarrow 0} \sin(a+h)}{\lim_{h \rightarrow 0} \cos(a+h)} \\
&= \frac{\lim_{h \rightarrow 0} (\sin a \cos h + \cos a \sin h)}{\lim_{h \rightarrow 0} (\cos a \cos h - \sin a \sin h)} \\
&= \left(\frac{\sin a \times \cos 0 + \cos a \times \sin 0}{\cos a \times \cos 0 - \sin a \times \sin 0} \right) = \left(\frac{\sin a \times 1 + \cos a \times 0}{\cos a \times 1 - \sin a \times 0} \right) \\
&= \tan a.
\end{aligned}$$

Thus, $\tan x$ is continuous at a for all $a \in \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2} : n \in \mathbb{I} \right\}$.

Hence, $\tan x$ is continuous.

THEOREM 10 *The logarithmic function is continuous.*

PROOF Let $f(x) = \log x$. Then, $\text{dom}(f)$ = set of positive real numbers.

Let a be any positive real number. Then,

$$\begin{aligned}
\lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} \log x \\
&= \lim_{x \rightarrow a} \left[\log \left(a \cdot \frac{x}{a} \right) \right] = \lim_{x \rightarrow a} \left[\log a + \log \frac{x}{a} \right] \\
&= \log a + \lim_{x \rightarrow a} \log \frac{x}{a} = \log a = f(a) \left[\because \lim_{x \rightarrow 0} \log \frac{x}{a} = \log 1 = 0 \right].
\end{aligned}$$

Thus, $f(x) = \log x$ is continuous at $a \in \mathbb{R}^+$.

Hence, $f(x) = \log x$ is continuous.

THEOREM 11 *$\sin |x|$ is continuous.*

PROOF Let $f(x) = |x|$ and $g(x) = \sin x$. Then,

$$(g \circ f)(x) = g\{f(x)\} = g(|x|) = \sin |x|.$$

Now, f and g being continuous, it follows that their composite $(g \circ f)$ is continuous.

Hence, $\sin |x|$ is continuous.

SOLVED EXAMPLES

EXAMPLE 1 Let $f(x) = \begin{cases} x & \text{if } x \geq 1 \\ x^2 & \text{if } x < 1. \end{cases}$ Is f a continuous function? Why?

SOLUTION Let a be any real number. Then, three possibilities arise.

Case I When $a > 1$

In this case, $f(a) = a$.

$$\text{Also, } \lim_{x \rightarrow a+} f(x) = \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} (a+h) = a.$$

$$\text{And, } \lim_{x \rightarrow a-} f(x) = \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} (a-h) = a$$

$$[\because a > 1 \text{ and } h \text{ is very small} \Rightarrow (a-h) > 1].$$

$$\therefore \lim_{x \rightarrow a+} f(x) = \lim_{a \rightarrow a-} f(x) = a = f(a).$$

So, $f(x)$ is continuous at each $a > 1$.

Case II When $a < 1$

In this case, $f(a) = a^2$.

$$\lim_{x \rightarrow a+} f(x) = \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} (a+h)^2 = a^2$$

$$[\because a < 1 \text{ and } h \text{ is very small} \Rightarrow (a+h) < 1].$$

$$\text{And, } \lim_{x \rightarrow a-} f(x) = \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} (a-h)^2 = a^2$$

$$[\because a < 1 \text{ and } h \text{ is very small} \Rightarrow (a-h) < 1].$$

$$\therefore \lim_{x \rightarrow a+} f(x) = \lim_{x \rightarrow a-} f(x) = a^2 = f(a).$$

Case III When $a = 1$

In this case, $f(1) = 1$.

$$\lim_{x \rightarrow 1+} f(x) = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} (1+h) = 1 \quad [\because (1+h) > 1].$$

$$\lim_{x \rightarrow 1-} f(x) = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} (1-h)^2 = 1 \quad [\because (1-h) < 1].$$

$$\therefore \lim_{x \rightarrow 1+} f(x) = \lim_{x \rightarrow 1-} f(x) = 1.$$

$$\text{So, } \lim_{x \rightarrow 1} f(x) = 1 = f(1).$$

So, f is continuous at $a = 1$.

Thus, from all the above three cases, it follows that $f(x)$ is continuous at $x = a$ for all $a \in \mathbb{R}$.

Hence, $f(x)$ is continuous.

EXAMPLE 2 Prove that $f(x) = |x|$ is a continuous function.

SOLUTION Let $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$

Clearly, $\text{dom}(f) = \mathbb{R}$.

Let a be any real number. Then, the following cases arise.

Case I When $a < 0$

In this case, $f(a) = -a$.

$$\lim_{x \rightarrow a+} f(x) = \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} \{-(a+h)\} = -a$$

$$[\because a < 0 \text{ and } h \text{ is very small and positive} \Rightarrow a+h < 0].$$

$$\text{And, } \lim_{x \rightarrow a-} f(x) = \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} \{-(a-h)\} = -a.$$

$$[\because a < 0 \text{ and } h \text{ is very small and positive} \Rightarrow a-h < 0].$$

$$\text{So, } \lim_{x \rightarrow a+} f(x) = \lim_{x \rightarrow a-} f(x) = -a = f(a).$$

Thus, $f(x)$ is continuous at each $a < 0$.

Case II When $a > 0$

In this case, $f(a) = a$.

$$\lim_{x \rightarrow a+} f(x) = \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} (a+h) = a \quad [\because (a+h) > 0].$$

$$\lim_{x \rightarrow a-} f(x) = \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} (a-h) = a \quad [\because (a-h) > 0].$$

$$\therefore \lim_{x \rightarrow a+} f(x) = \lim_{x \rightarrow a-} f(x) = f(a).$$

So, $f(x)$ is continuous at each $a > 0$.

Case III When $a = 0$

Clearly, $f(0) = 0$.

$$\lim_{x \rightarrow 0+} f(x) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} h = 0.$$

$$\begin{aligned} \lim_{x \rightarrow 0-} f(x) &= \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} \{-(-h)\} \\ &= \lim_{h \rightarrow 0} h = 0. \end{aligned}$$

$$\therefore \lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0-} f(x) = 0 = f(0).$$

So, $f(x)$ is continuous at $x = 0$.

Thus, from all the above cases, it follows that $f(x) = |x|$ is continuous at a for all $a \in \mathbb{R}$.

Hence, $f(x) = |x|$ is continuous.

EXAMPLE 3 Discuss the continuity of the function $f(x) = \begin{cases} 2x-1, & \text{if } x < 0 \\ 2x+1, & \text{if } x \geq 0. \end{cases}$ [CBSE 2002]

SOLUTION When $x < 0$, we have $f(x) = 2x-1$, which being a polynomial function, is continuous at each point where $x < 0$.

Also, when $x > 0$, we have $f(x) = 2x+1$, which being a polynomial function, is continuous at each point where $x > 0$.

Let us consider the point $x = 0$.

$$f(0) = (2 \times 0 + 1) = 1.$$

$$\lim_{x \rightarrow 0+} f(x) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} (2h+1) = 1.$$

$$\text{And, } \lim_{x \rightarrow 0-} f(x) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} [(2(-h)-1)] = -1.$$

$$\therefore \lim_{x \rightarrow 0+} f(x) \neq \lim_{x \rightarrow 0-} f(x).$$

So, $\lim_{x \rightarrow 0} f(x)$ does not exist, and therefore $f(x)$ is not continuous at $x = 0$.

Thus, the given function is continuous at each point except at $x = 0$, where it is discontinuous.

EXAMPLE 4 Discuss the continuity of the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ (x+1), & \text{if } x \geq 0. \end{cases}$

SOLUTION We know that $\sin x$ as well as the identity function x is continuous.

So, the quotient function, $\frac{\sin x}{x}$ is continuous at each $x < 0$.

Also, when $x > 0$, we have $f(x) = (x + 1)$, which being a polynomial function, is continuous.

Let us consider the point $x = 0$.

Clearly, $f(0) = (0 + 1) = 1$.

We have $f(0) = (0 + 1) = 1$;

$$\lim_{x \rightarrow 0+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} (h + 1) = 1$$

$$\begin{aligned} \text{and, } \lim_{x \rightarrow 0-} f(x) &= \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h) \\ &= \lim_{h \rightarrow 0} \frac{\sin(-h)}{-h} = \lim_{h \rightarrow 0} \left(\frac{-\sin h}{-h} \right) = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1. \end{aligned}$$

$$\therefore \lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0-} f(x) = 1.$$

$$\text{So, } \lim_{x \rightarrow 0} f(x) = 1 = f(0).$$

Thus, $f(x)$ is continuous at $x = 0$ also.

Hence, $f(x)$ is continuous at all points.

EXAMPLE 5 Discuss the continuity of the function $f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

SOLUTION We know that the identity function x is continuous and the modulus function $|x|$ is continuous.

So, the quotient function $\frac{x}{|x|}$ is continuous at each $x \neq 0$.

It has already been proved that $f(x)$ is discontinuous at $x = 0$.

Hence, the given function is continuous at each point, except at $x = 0$.

EXAMPLE 6 Locate the point of discontinuity of the function $f(x) = \begin{cases} \frac{x^4 - 16}{x - 2}, & \text{if } x \neq 2 \\ 16, & \text{if } x = 2. \end{cases}$

SOLUTION The function $f(x) = \frac{x^4 - 16}{x - 2}$ being a rational function, is continuous at all points of its domain, i.e., for all real numbers except 2.

Now, consider the given function at $x = 2$.

We have $f(2) = 16$.

$$\begin{aligned} \lim_{x \rightarrow 2} f(x) &= \lim_{h \rightarrow 0} f(2 + h) = \lim_{h \rightarrow 0} \left\{ \frac{(2 + h)^4 - 16}{2 + h - 2} \right\} = \lim_{2+h \rightarrow 2} \left\{ \frac{(2 + h)^4 - 2^4}{(2 + h) - 2} \right\} \\ &= 4 \times 2^{4-1} = 32 \quad \left[\because \lim_{x \rightarrow a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right]. \end{aligned}$$

Thus, $\lim_{x \rightarrow 2+} f(x) \neq f(2)$, and therefore $\lim_{x \rightarrow 2} f(x) \neq f(2)$.

Hence, $f(x)$ is discontinuous at $x = 2$.

EXAMPLE 7 Determine the value of k so that the function

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2; \\ 3, & \text{if } x > 2 \end{cases} \text{ is continuous.} \quad [\text{CBSE 1990C, '91}]$$

SOLUTION Since a polynomial function is continuous and a constant function is continuous, the given function is continuous for all $x < 2$ and for all $x > 2$.

So, consider the point $x = 2$. We have $f(2) = 4k$.

$$\lim_{x \rightarrow 2+} f(x) = \lim_{h \rightarrow 0} f(2+h) = \lim_{h \rightarrow 0} 3 = 3.$$

$$\text{And, } \lim_{x \rightarrow 2-} f(x) = \lim_{h \rightarrow 0} f(2-h) = \lim_{h \rightarrow 0} k(2-h)^2 = 4k.$$

$$\therefore \text{ for continuity, we must have } 4k = 3 \quad \text{or} \quad k = \frac{3}{4}.$$

EXAMPLE 8 Let $f(x) = \begin{cases} 1, & \text{if } x \leq 3 \\ ax + b, & \text{if } 3 < x < 5 \\ 7, & \text{if } 5 \leq x. \end{cases}$

Find the values of a and b so that $f(x)$ is continuous.

SOLUTION We know that a constant function is continuous, and a polynomial function is continuous.

So, the given function is continuous for all $x < 3$; for all $x > 5$ and for all x lying in $]3, 5[$.

Now, consider the point $x = 3$. We have $f(3) = 1$.

$$\lim_{x \rightarrow 3+} f(x) = \lim_{h \rightarrow 0} f(3+h) = \lim_{h \rightarrow 0} [a(3+h) + b] = (3a + b).$$

$$\text{And, } \lim_{x \rightarrow 3-} f(x) = \lim_{h \rightarrow 0} (3-h) = \lim_{h \rightarrow 0} 1 = 1.$$

Since $f(x)$ is given to be continuous, it must be continuous at $x = 3$ also.

$$\text{So, we must have } f(3) = \lim_{x \rightarrow 3-} f(x) = \lim_{x \rightarrow 3+} f(x), \text{ i.e., } 3a + b = 1. \dots (i)$$

Again, consider the point $x = 5$. We have $f(5) = 7$.

$$\lim_{x \rightarrow 5+} f(x) = \lim_{h \rightarrow 0} f(5+h) = \lim_{h \rightarrow 0} 7 = 7.$$

$$\text{And, } \lim_{x \rightarrow 5-} f(x) = \lim_{h \rightarrow 0} f(5-h) = \lim_{h \rightarrow 0} [a(5-h) + b] = (5a + b).$$

Now, by continuity of $f(x)$ at $x = 5$, we have

$$f(5) = \lim_{x \rightarrow 5+} f(x) = \lim_{x \rightarrow 5-} f(x) \quad \text{or} \quad 5a + b = 7. \dots (ii)$$

Solving (i) and (ii), we get $a = 3$ and $b = -8$.

EXAMPLE 9 Show that the function $f(x) = \sqrt{x^4 + 3}$ is continuous at each point.

SOLUTION Let $g(x) = x^4 + 3$ and $h(y) = \sqrt{y}$.

Since every polynomial function is continuous at each point where $x = a$, it follows that $g(x)$ is continuous at $x = a$.

Clearly, $g(a)$ is positive. Moreover, $h[g(a)] = \sqrt{g(a)} = \lim_{y \rightarrow g(a)} \sqrt{y}$.

$\therefore h(y)$ is continuous at $g(a)$.

Since the composite of continuous functions is continuous, it follows that $(h \circ g)(x)$ is continuous.

But, $(h \circ g)(x) = h[g(x)] = h(x^4 + 3) = \sqrt{x^4 + 3} = f(x)$.

Hence, $f(x)$ is continuous.

EXAMPLE 10 Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

SOLUTION Let $g(x) = \sin x + \cos x$ and $k(y) = |y|$.

We shall first show that g is continuous at $x = \pi$ and k is continuous at $y = g(\pi)$.

Now, $\lim_{x \rightarrow \pi} g(x) = \lim_{x \rightarrow \pi} (\sin x + \cos x) = \sin \pi + \cos \pi = -1$.

Also, $g(\pi) = (\sin \pi + \cos \pi) = -1$. $\therefore \lim_{x \rightarrow \pi} g(x) = g(\pi)$.

So, g is continuous at $x = \pi$. Now, $g(\pi) = -1$.

$\therefore k[g(\pi)] = k(-1) = |-1| = 1$.

Now, $\lim_{y \rightarrow (-1)^+} k(y) = \lim_{y \rightarrow (-1)^+} |y| = \lim_{h \rightarrow 0} |-1 + h| = 1$.

And, $\lim_{y \rightarrow (-1)^-} k(y) = \lim_{y \rightarrow (-1)^-} |y| = \lim_{h \rightarrow 0} |-1 - h| = 1$.

$\therefore k[g(\pi)] = \lim_{y \rightarrow g(\pi)} k(y)$. This shows that k is continuous at $g(\pi)$.

Consequently, $(k \circ g)$ is continuous at $x = \pi$.

But, $(k \circ g)(x) = k[g(x)] = k(\sin x + \cos x) = |\sin x + \cos x| = f(x)$.

Hence, $f(x)$ is continuous at $x = \pi$.

EXERCISE 9B

1. Show that the function $f(x) = \begin{cases} (7x+5), & \text{when } x \geq 0; \\ (5-3x), & \text{when } x < 0 \end{cases}$ is a continuous function.
2. Show that the function $f(x) = \begin{cases} \sin x, & \text{if } x < 0; \\ x, & \text{if } x \geq 0 \end{cases}$ is continuous.
3. Show that the function $f(x) = \begin{cases} \frac{x^n - 1}{x - 1}, & \text{when } x \neq 1; \\ n, & \text{when } x = 1 \end{cases}$ is continuous.
4. Show that $\sec x$ is a continuous function.
5. Show that $\cos |x|$ is a continuous function.

6. Show that the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{when } x \neq 0; \\ 2, & \text{when } x = 0 \end{cases}$ is continuous at each point except 0.
7. Discuss the continuity of $f(x) = [x]$.
8. Show that $f(x) = \begin{cases} (2x-1), & \text{if } x < 2; \\ \frac{3x}{2}, & \text{if } x \geq 2 \end{cases}$ is continuous.
9. Show that $f(x) = \begin{cases} x, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at each point except 0.
10. Locate the point of discontinuity of the function $f(x) = \begin{cases} (x^3 - x^2 + 2x - 2), & \text{if } x \neq 1; \\ 4, & \text{if } x = 1. \end{cases}$
11. Discuss the continuity of the function $f(x) = |x| + |x-1|$ in the interval $[-1, 2]$.

ANSWERS (EXERCISE 9B)

7. discontinuous at $x = n$, where n is an integer
10. discontinuous at $x = 1$ 11. continuous

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 9B)

1. Since a polynomial function is continuous, the given function is continuous at each $x > 0$ as well as at each $x < 0$. Test its continuity at $x = 0$.
2. Let $a \in \mathbb{R}$.
- Case I When $a > 0$ then, $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a) = a$.
- Case II When $a < 0$ then, $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a) = \sin a$.
- Case III When $a = 0$ then, $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0) = 0$.
7. $f(x)$ is discontinuous at each integral value of x .
8. Take three cases: (i) $a > 2$ (ii) $a < 2$ (iii) $a = 2$.
10. Test at $x = 1$.
11. The given function may be expressed as

$$f(x) = \begin{cases} (1-2x), & \text{when } -1 \leq x < 0; \\ 1, & \text{when } 0 \leq x \leq 1; \\ (2x-1), & \text{when } 1 < x \leq 2. \end{cases}$$

Since a polynomial function as well as a constant function is continuous, it follows that $f(x)$ is continuous when $-1 < x < 0$; $0 < x < 1$ and $1 < x < 2$.

So, test the continuity at $-1, 0, 1$ and 2 .

At $x = -1$, $\lim_{x \rightarrow (-1)^+} f(x) = f(-1)$, so it is continuous at $x = -1$.

At $x = 0$, $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0) = 1$, so it is continuous at $x = 0$.

At $x = 1$, $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1) = 1$.

At $x = 2$, $\lim_{x \rightarrow 2^-} f(x) = f(2) = 3$, so it is continuous at $x = 2$.

Differentiability

Let $f(x)$ be a real function and a be any real number. Then, we define

(i) **Right-hand derivative** $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$, and it is denoted by $Rf'(a)$.

(ii) **Left-hand derivative** $\lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$, and it is denoted by $Lf'(a)$.

DIFFERENTIABILITY A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.

The common value of $Rf'(a)$ and $Lf'(a)$ is denoted by $f'(a)$ and it is known as the *derivative* of $f(x)$ at $x = a$.

If, however, $Rf'(a) \neq Lf'(a)$, we say that $f(x)$ is not differentiable at $x = a$.

REMARK In each case, h is taken as positive and very small.

SOLVED EXAMPLES

EXAMPLE 1 Show that $f(x) = x^2$ is differentiable at $x = 1$ and find $f'(1)$.

SOLUTION $Rf'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{(1+h)^2 - (1)^2}{h}$
 $= \lim_{h \rightarrow 0} \left(\frac{1 + h^2 + 2h - 1}{h} \right) = \lim_{h \rightarrow 0} (h + 2) = 2.$
 And, $Lf'(1) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \rightarrow 0} \frac{(1-h)^2 - (1)^2}{-h}$
 $= \lim_{h \rightarrow 0} \left(\frac{1 + h^2 - 2h - 1}{-h} \right) = \lim_{h \rightarrow 0} (-h + 2) = 2.$
 $\therefore Rf'(1) = Lf'(1) = 2.$

This shows that $f(x)$ is differentiable at $x = 1$ and $f'(1) = 2$.

EXAMPLE 2 Show that $f(x) = [x]$ is not differentiable at $x = 1$.

SOLUTION We have $Rf'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{[1+h] - [1]}{h} = 0$
 $\{\because [1+h] = 1 \text{ and } [1] = 1\},$

$$\text{and } Lf'(1) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \rightarrow 0} \frac{[1-h] - [1]}{-h} = \infty$$

$$\{\because [1-h] = 0 \text{ and } [1] = 1\}.$$

Thus, $Rf'(1) \neq Lf'(1)$.

Hence, $f(x) = [x]$ is not differentiable at $x = 1$.

EXAMPLE 3 (i) Show that $f(x) = x^{4/3}$ is differentiable at $x = 0$, and hence find $f'(0)$.
(ii) Show that $g(x) = x^{3/2}$ is not differentiable at $x = 0$.

SOLUTION (i) We have $Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h \rightarrow 0} \frac{h^{4/3} - 0}{h} = \lim_{h \rightarrow 0} \frac{h^{4/3}}{h} = \lim_{h \rightarrow 0} h^{1/3} = 0.$$

And, $Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{f(-h) - 0}{-h}$

$$= \lim_{h \rightarrow 0} \frac{(-h)^{4/3} - 0}{(-h)} = \lim_{h \rightarrow 0} \frac{(-h)^{4/3}}{(-h)} = \lim_{h \rightarrow 0} (-h)^{1/3} = 0.$$

Thus, $Rf'(0) = Lf'(0) = 0$.

Hence, $f(x) = x^{4/3}$ is differentiable at $x = 0$ and $f'(0) = 0$.

(ii) Consider $g(x) = x^{3/2}$.

Now, $Rg'(0) = \lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h}$

$$= \lim_{h \rightarrow 0} \frac{h^{3/2} - 0}{h} = \lim_{h \rightarrow 0} \frac{h^{3/2}}{h} = \lim_{h \rightarrow 0} h^{1/2} = 0.$$

And, $Lg'(0) = \lim_{h \rightarrow 0} \frac{g(0-h) - g(0)}{-h} = \lim_{h \rightarrow 0} \frac{g(-h) - g(0)}{-h}$

$$= \lim_{h \rightarrow 0} \frac{(-h)^{3/2} - 0}{(-h)} = \lim_{h \rightarrow 0} \frac{(-h)^{3/2}}{(-h)}$$

$$= \lim_{h \rightarrow 0} (-h)^{1/2}, \text{ which is imaginary.}$$

Thus, $Lg'(0)$ does not exist.

Hence, $g(x) = x^{3/2}$ is not differentiable at $x = 0$.

EXAMPLE 4 Show that the function $f(x) = \begin{cases} 1+x, & \text{if } x \leq 2; \\ 5-x, & \text{if } x > 2 \end{cases}$ is not differentiable at $x = 2$.

SOLUTION $Rf'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \left[\frac{5 - (2+h) - 3}{h} \right]$

$$= \lim_{h \rightarrow 0} \frac{-h}{h} = \lim_{h \rightarrow 0} (-1) = -1.$$

$$\begin{aligned}\text{And, } Lf'(2) &= \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h} \\ &= \lim_{h \rightarrow 0} \left[\frac{1 + (2-h) - 3}{-h} \right] = \lim_{h \rightarrow 0} \frac{-h}{-h} = \lim_{h \rightarrow 0} 1 = 1.\end{aligned}$$

Thus, $Rf'(2) \neq Lf'(2)$.

Hence, $f(x)$ is not differentiable at $x = 2$.

EXAMPLE 5 Let $f(x) = \begin{cases} (1 + \sin x), & \text{when } 0 \leq x < \frac{\pi}{2} \\ 1, & \text{when } x < 0. \end{cases}$

Show that $f'(0)$ does not exist.

SOLUTION We have $Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h \rightarrow 0} \frac{(1 + \sin h) - 1}{h} = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1.$$

$$\text{And, } Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{(1-1)}{-h} = 0.$$

Thus, $Rf'(0) \neq Lf'(0)$. Hence, $f'(0)$ does not exist.

EXAMPLE 6 Let $f(x) = mx + c$ and $f(0) = f'(0) = 1$. Find $f(2)$.

SOLUTION Clearly, $f(0) = (m \times 0 + c) = c = 1$ (given).

$$\begin{aligned}\text{Also, } f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(mh + c) - 1}{h} = \lim_{h \rightarrow 0} \frac{mh + 1 - 1}{h} \quad [\because c = 1] \\ &= \lim_{h \rightarrow 0} \frac{mh}{h} = \lim_{h \rightarrow 0} m = m.\end{aligned}$$

$$\text{Thus, } f'(0) = 1 \Rightarrow m = 1.$$

$$\text{So, } f(x) = 1 \cdot x + 1, \text{ i.e., } f(x) = (x + 1). \text{ Hence, } f(2) = (2 + 1) = 3.$$

Relation between Continuity and Differentiability

THEOREM Every differentiable function is continuous. But, every continuous function need not be differentiable.

PROOF Let $f(x)$ be a differentiable function and let a be any real number in its domain. Then, $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a)$ (i)

$$\begin{aligned}\text{Now, } \lim_{h \rightarrow 0} [f(a+h) - f(a)] &= \lim_{h \rightarrow 0} \left[\frac{f(a+h) - f(a)}{h} \times h \right] \\ &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \times \lim_{h \rightarrow 0} h \\ &= f'(a) \times 0 = 0 \quad [\text{using (i)}].\end{aligned}$$

$$\text{Thus, } \lim_{h \rightarrow 0} [f(a+h) - f(a)] = 0 \quad \text{or} \quad \lim_{h \rightarrow 0} f(a+h) = f(a).$$

This shows that $f(x)$ is continuous at a for all a .

Hence, every differentiable function is continuous.

In order to show that a continuous function need not be differentiable, it is sufficient to give an example of a function which is continuous but not differentiable. Consider $f(x) = |x|$ at $x = 0$.

Clearly, $f(0) = 0$.

$$\lim_{x \rightarrow 0+} f(x) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} |h| = \lim_{h \rightarrow 0} h = 0.$$

$$\text{And, } \lim_{x \rightarrow 0-} f(x) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} |-h| = \lim_{h \rightarrow 0} h = 0.$$

$$\text{Thus, } \lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0-} f(x) = f(0).$$

So, $f(x) = |x|$ is continuous at $x = 0$.

$$\begin{aligned} \text{But, } Rf'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1. \end{aligned}$$

$$\begin{aligned} \text{And, } Lf'(0) &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{|-h| - 0}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = -1. \end{aligned}$$

Thus, $Rf'(0) \neq Lf'(0)$.

This shows that $f(x) = |x|$ is not differentiable at $x = 0$. Thus, $f(x) = |x|$ is continuous but not differentiable at $x = 0$. Hence, a continuous function need not be differentiable.

EXAMPLE 7 Show that the function $f(x) = \begin{cases} x \sin(1/x), & \text{when } x \neq 0; \\ 0, & \text{when } x = 0 \end{cases}$

is continuous but not differentiable at $x = 0$.

SOLUTION We have already discussed the above function for continuity at $x = 0$.

$$\begin{aligned} \text{Now, } Rf'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h \sin(1/h) - 0}{h} \\ &= \lim_{h \rightarrow 0} \sin(1/h), \text{ which does not exist.} \end{aligned}$$

Hence, $f(x)$ is not differentiable at $x = 0$.

EXAMPLE 8 Show that $f(x) = |x - 2|$ is continuous but not differentiable at $x = 2$.

SOLUTION We have $f(2) = |2 - 2| = 0$.

$$\lim_{x \rightarrow 2+} f(x) = \lim_{h \rightarrow 0} f(2+h) = \lim_{h \rightarrow 0} |2+h-2| = \lim_{h \rightarrow 0} |h| = \lim_{h \rightarrow 0} h = 0.$$

$$\lim_{x \rightarrow 2-} f(x) = \lim_{h \rightarrow 0} f(2-h) = \lim_{h \rightarrow 0} |2-h-2| = \lim_{h \rightarrow 0} |-h| = \lim_{h \rightarrow 0} h = 0.$$

$$\therefore \lim_{x \rightarrow 2+} f(x) = \lim_{x \rightarrow 2-} f(x) = f(2) = 0.$$

So, $f(x)$ is continuous at $x = 2$.

$$\begin{aligned}\text{But, } Rf'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{|2+h-2| - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.\end{aligned}$$

$$\begin{aligned}\text{And, } Lf'(2) &= \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \rightarrow 0} \frac{|2-h-2| - 0}{-h} \\ &= \lim_{h \rightarrow 0} \frac{|-h|}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = -1.\end{aligned}$$

Thus, $Rf'(2) \neq Lf'(2)$.

This shows that $f(x)$ is not differentiable at $x = 2$.

EXERCISE 9C

1. Show that $f(x) = x^3$ is continuous as well as differentiable at $x = 3$.
2. Show that $f(x) = (x-1)^{1/3}$ is not differentiable at $x = 1$.
3. Show that a constant function is always differentiable.
4. Show that $f(x) = |x-5|$ is continuous but not differentiable at $x = 5$.
5. Let $f(x) = \begin{cases} (2-x), & \text{when } x \geq 1 \\ x, & \text{when } 0 \leq x \leq 1. \end{cases}$

Show that $f(x)$ is continuous but not differentiable at $x = 1$.

6. Show that $f(x) = [x]$ is neither continuous nor derivable at $x = 2$.

7. Show that the function $f(x) = \begin{cases} (1-x), & \text{when } x < 1; \\ (x^2-1), & \text{when } x \geq 1 \end{cases}$

is continuous but not differentiable at $x = 1$.

8. Let $f(x) = \begin{cases} (2+x), & \text{if } x \geq 0 \\ (2-x), & \text{if } x < 0. \end{cases}$ Show that $f(x)$ is not derivable at $x = 0$.

9. If $f(x) = |x|$, show that $f'(2) = 1$.

10. Find the values of a and b so that the function

$$f(x) = \begin{cases} (x^2 + 3x + a), & \text{when } x \leq 1; \\ (bx + 2), & \text{when } x > 1 \end{cases} \text{ is differentiable at each } x \in R.$$

ANSWERS (EXERCISE 9C)

10. $a = 3, b = 5$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 9C)

9. Show that $Rf'(2) = Lf'(2) = 1$.
10. $Rf'(1) = Lf'(1) = 5$.